8.3 Autonomous Differential Equations and Stability

Logistic Growth Model

1. **Exercise: Bacteria Growth.** Many bacteria strains are used by the dairy industry to produce different types of fermented milks and yogurts. During a fermentation experiment, the population $y$ (in millions per mL) of *Lactobacillus fermentum* after $t$ hours in a wheat medium satisfied the differential equation

$$ y' = 0.532y \left(1 - \frac{y}{1900}\right). $$

Initially, 8 million bacteria per mL were present. Determine the number of bacteria per mL after 5 hr.

8.4 Separable Differential Equations

Explaining separable differential equations with the following first-order initial value problem

$$ 3y^2 \frac{dy}{dx} + x = 0, \quad y(4) = 2 $$

2. **Exercise:** Solve the differential equation $\frac{dy}{dx} = 3x^2e^{2y}$.

Implicitly Defined Solutions

3. **Exercise:** Find the solution of the initial-value problem $y' = \frac{2x}{y+e^{5y}}, \quad y(2) = 0$.

8.4 Numerical Solutions of Differential Equations

**Euler’s Method**

There is a requirement to use methods for generating approximate solutions for the given equation. This situation is analogous to the evaluation of definite integrals. There are some important definite integrals that cannot be exactly evaluated using any technique. However, there are some numerical methods to approximate the values of such integrals. Now, here a discussion will be conducted to develop methods of numerically solving the initial-value problem

$$ y' = f(x,y), \quad y(x_0) = y_0 $$

These techniques are called **Euler’s method** and the **Runge-Kutta method**.

**Euler’s Method**

First choose a distance $\Delta x$, called the step size. Then we define the sequence of points,

$$ x_1 = x_0 + \Delta x, $$

$$ x_2 = x_1 + \Delta x = x_0 + 2\Delta x, $$
\[ x_3 = x_2 + \Delta x = x_0 + 3\Delta x, \quad \text{and so on} \]

Euler’s method will approximate the exact solution only at these points.

**Definition:**
To approximate the solution of (1) using Euler’s method, we use the formula

\[ y_{n+1} = y_n + f(x_n, y_n) \Delta x, \]

where \( y_n \approx y(x_n) \)

4. **Exercise:** Use Euler’s method with \( \Delta x = 0.2 \) to approximate the solution of \( y' = (4x - 1)y, \quad y(0) = 2 \) on the interval \([0,1]\).

**Runge-Kutta Method**

**Definition:**
To approximate the solution of the initial-value problem

\[ y' = f(x, y), \quad y(x_0) = y_0 \]

Using the **Runge-Kutta method**, we use the following five formulas in order:

\[
\begin{align*}
    a_n &= f(x_n, y_n) \\
    b_n &= f(x_n + \frac{\Delta x}{2}, y_n + a_n \frac{\Delta x}{2}) \\
    c_n &= f(x_n + \frac{\Delta x}{2}, y_n + b_n \frac{\Delta x}{2}) \\
    d_n &= f(x_n + \Delta x, y_n + c_n \Delta x) \\
    y_{n+1} &= y_n + \frac{\Delta x}{6} (a_n + 2b_n + 2c_n + d_n)
\end{align*}
\]

Once again, \( \Delta x \) is the step size and \( y_n \) approximates \( y(x_n) \).

5. **Exercise:** Use the Runge-Kutta method with \( \Delta x = 0.2 \) to approximate the solution of \( y' = (4x - 1)y, \quad y(0) = 2 \) on the interval \([0, 1]\).