Constructing Complicated Spheres as Test Examples for Homology Algorithms

Mimi Tsuruga
Supervisors: Frank Lutz, John Sullivan

Berlin Mathematical School
TU Berlin, Germany

mimi.tsuruga@fu-berlin.de

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Outline

1. Homology Algorithms
   - Homology
   - Discrete Morse Theory
2. Akbulut-Kirby Spheres
   - Background
   - Construction
3. Mazur's 4-manifold

What is Homology?

- Homology is a basic tool for classifying topological spaces.
- For a given topological space X $\Leftarrow$ (simplicial complex) define chain complex $C(X)$ encoding information about X. $C(X)$ is an exact sequence of abelian groups C_i connected by homomorphisms $\partial_i : C_i \to C_{i-1}$. $\Leftarrow$ boundary maps

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Then the n-th homology group is the quotient group

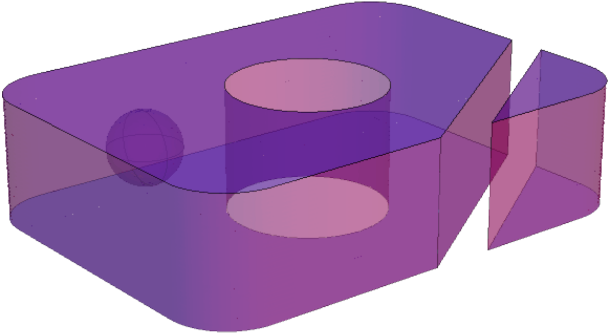
$$H_n := \mathrm{ker}\partial_n / \mathrm{im}\partial_{n+1}$$

- Fundamental theorem of finitely generated Abelian groups:

$$H_n \cong \mathbb{Z}^{\beta_n} \times \mathbb{Z}_{({p_1}^{r_1})} \times \mathbb{Z}_{({p_2}^{r_2})} \times \cdots \times \mathbb{Z}_{({p_m}^{r_m})}$$

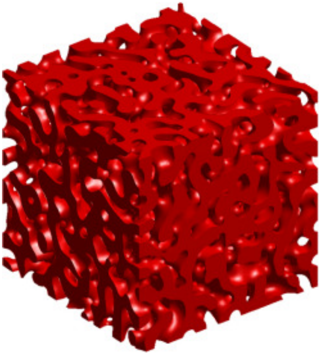
Counting Holes

- The Betti numbers β_i counts "holes".



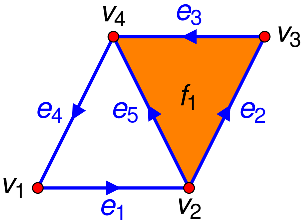
$\beta_0 = 2 \qquad \beta_1 = 1 \qquad \beta_2 = 1$

Applications



[source: Mischaikow, et. al.]

An Example.



C_0

∂_1		v_1	v_2	v_3	v_4
C_1	e_1	-1	1	0	0
	e_2	0	-1	1	0
	e_3	0	0	-1	1
	e_4	1	0	0	-1
	e_5	0	-1	0	1

C_1

∂_2		e_1	e_2	e_3	e_4	e_5
C_2	f_1	0	1	1	0	-1

$$\beta_1 = \text{rank}\,(H_1) = \text{rank}\,(\ \ker \partial_1 \ /\ \text{im}\ \partial_2\ )$$

An Example.

$$A_1 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & -1 \end{pmatrix}$$

$$\begin{aligned} \beta_1 = \mathrm{rank}(H_1) &= \mathrm{rank}(\ker\partial_1/\mathrm{im}\partial_2) \\ &= \ker A_1 - \mathrm{rk} A_2 \\ &= 2 - 1 = 1 \end{aligned}$$

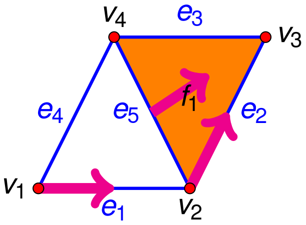
Smith Normal Form

- Computing the Smith Normal Form is polynomial.
- In practice, complexes can have >1m cells!
- Preprocess using Discrete Morse Theory.
  (to reduce input complexes)

Definitions

Discrete Vector Field

A discrete vector field V on a simplicial complex K is a collection of disjoint pairs of (co-dimension 1) incident cells.



$$V = \{ (v_1 \prec e_1), (v_2 \prec e_3), (e_5 \prec f_1) \}$$

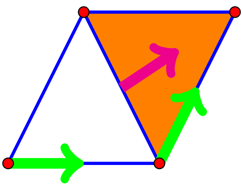
~~(v_1, e_4)~~

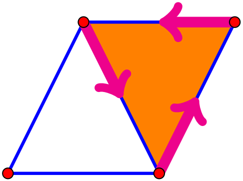
Definitions

V-path

A V-path is a subset of V that is a 'continuous' sequence of cells along which pairs of incident cells alternate being in and not in V.

A V-path is a cycle if this path 'closes'.





Definitions

Gradient Vector Field

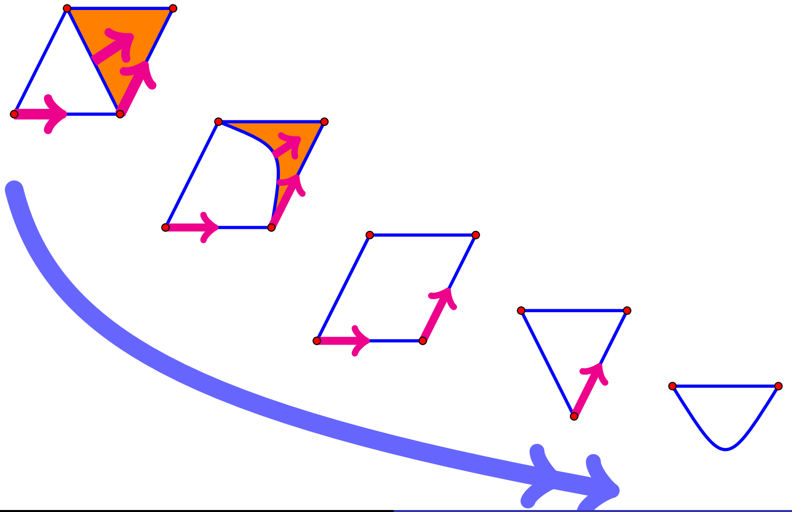
A discrete vector field V is a gradient vector field if there are no cycles.

Theorem

A gradient vector field on K corresponds to a discrete Morse function on K.

A discrete Morse function is a function $f : K \to \mathbb{R}$ with a certain property.

Collapse



Definitions

Critical Cell

A critical cell of a gradient vector field V on K is a cell of K which is not contained in any pair in V.

Discrete Morse Vector

A discrete Morse vector $c = (c_0, c_1, \ldots, c_d)$ is defined by

$$c_i = \#\text{ critical cells of dimension } 0 \leq i \leq d.$$

Morse Inequality

$$\beta_i \leq c_i \quad \forall i$$

Random Discrete Morse Theory

- Computing the optimal discrete Morse vectors is $\mathcal{NP}$-hard!

  [Lewiner, Lopes, Tavares, 2003], [Joswig, Pfetsch, 2006].

  $\Longrightarrow$ Use Random Discrete Morse Theory!
- Useful to determine the complicatedness of complexes.

  [Bendetti,Lutz],

SPC4

Poincaré Conjecture

Every simply connected closed 3-manifold is homeomorphic to the 3-sphere.

Generalized Poincaré Conjecture

Is a homotopy *d*-sphere homeomorphic (TOP)/diffeomorphic (DIFF)/PL-isomorphic (PL) to the *d*-sphere?

- 1960s Smale: dim=≥ 5
- 1982 Freedman: dim=4 (TOP)
- 2003 Perelman: dim=3

$\Longrightarrow$ Smooth Poincaré Conjecture in Dimension 4 (SPC4)

Exotic Spheres

1981 Akbulut, Kirby, "A potential smooth counterexample in dimension 4 to the Poincaré conjecture, ..." introduced Cappell-Shaneson spheres as potential counterexample for SPC4.

1989 Gompf, "Killing the Akubulut-Kirby Sphere ..."

The Basic Idea

- $\partial(\,(d+1)\text{-ball}\,) = S^d$.
- If you “fill" the hole of a donut $\Longrightarrow$ ball.
  In the case of d=3, use a solid cylinder. Need to specify the attaching map.
- Terminology:
  donut = ball with a 1-handle
  the solid cylinder used to close the donut = 2-handle

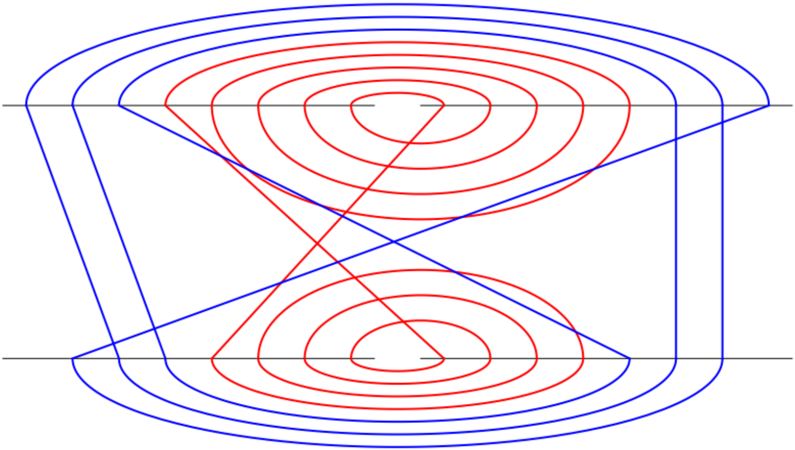
The Recipe

- Take a 5-ball.
- Add two 1-handles on it.
  Call them "x" and "y" to tell them apart.
- Attach two 2-handles to the surface of the 5 ball.

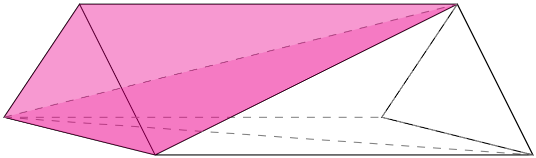
$$xyx = yxy \qquad x^r = y^{r-1}$$

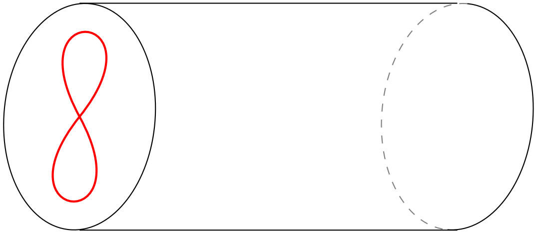
- Take the boundary.
- Done!

Spaghetti



Movin' On Up (In Dimension)





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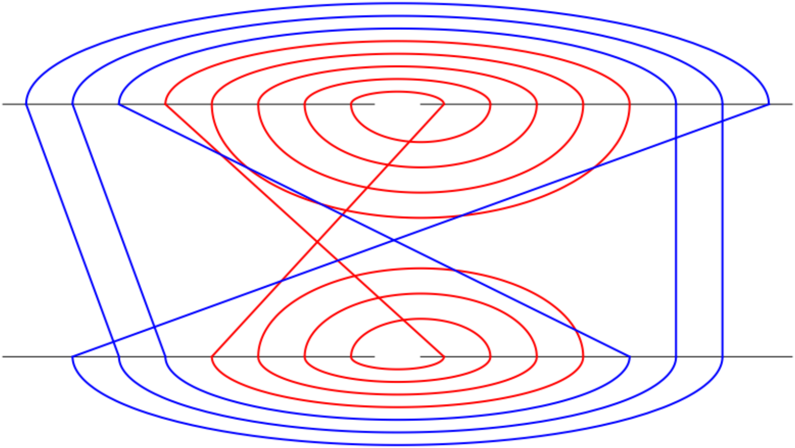

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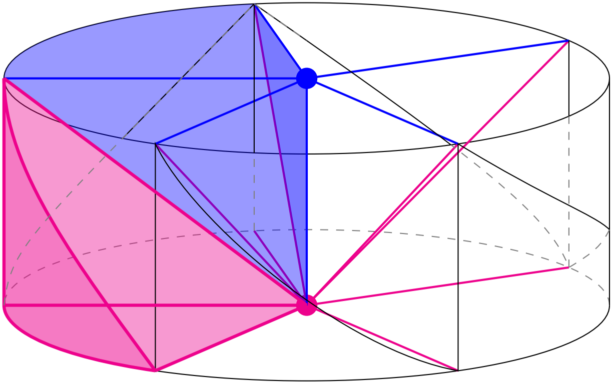
Handling 2-handles



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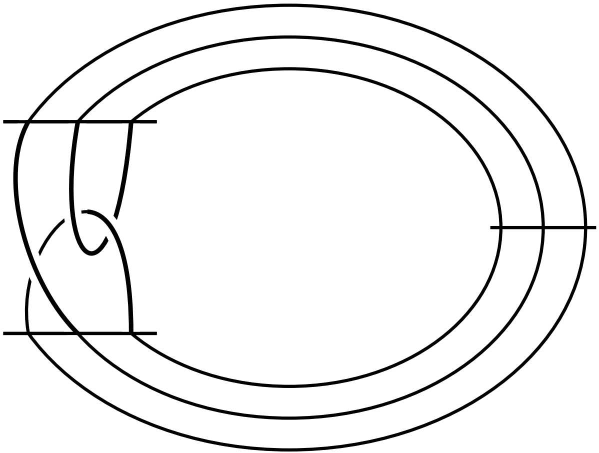
- Take the boundary.
- Done!

Results

- f-vector = (496, 7990, 27020, 32540, 13016)
- optimum discrete Morse vector = (1,2,4,2,1)
- Bistellar Flips $\Longrightarrow$ diffeomorphic
  - fundamental group with same presentation
  - further flips gave us trivial fundamental group
- Now being used to improve CHomP [Mischaikow,Nanda]

Construction



Results

- Contractible 4-manifold that is not a ball.
- f-vector = (103,992,2569,2569,890)
- Bistellar Flips on boundary $\Longrightarrow$ homotopy sphere $\Sigma(2,5,7)$
- Discrete Morse Spectrum:

$$[1,1,1,0,0] : 818$$
$$[1,2,2,0,0] : 158$$
$$[1,3,3,0,0] : 18$$
$$[1,4,4,0,0] : 4$$
$$[2,3,2,0,0] : 1$$
$$[2,4,3,0,0] : 1$$

Summary

- Homology algorithms use discrete Morse theory as a preprocessor.
- Random discrete Morse theory is used because finding optimum discrete Morse vector is NP-hard.
- Building complicated complexes to test algorithms.