Discrete Differential Geometry

Hunter College - Summer 2010
Mimi Tsuruga, Freie Universität

Lecture 5: July 1, 2010

8 Willmore Energy

Definition 8.1. The *discrete Willmore energy* at a vertex v is the sum

$$W(v) = \sum_{e \in V} \beta(e) - 2\pi$$

over all edges incident to v. The discrete Willmore energy of a compact simplicial surface S without boundary is the sum

$$W(S) = \frac{1}{2} \sum_{v \in V} W(v) = \sum_{e \in E} \beta(e) - \pi |V|$$

over all vertices V. This is Möbius invariant because they are circles and angles are preserved. In the planar case and if v and all its neighbors lie in S^2 (tangent plane to S^2 at v looks like planar case), $\sum \beta(e) = 2\pi$.

Definition 8.2. $S(v)$ is the *star* of all faces incident to v.

$$S(v) = \{f \in F \mid v \text{ incident to } f\}.$$

$S(v)$ is convex if $S(v)$ lies to one side of the plane of any of its faces.

Lemma 8.3. Let $\mathcal{P}$ be a closed (not necessarily planar) polygon in $\mathbb{R}^3$. Let β_i be its external angles. Take a point P and connect it to all vertices of polygon $\mathcal{P}$. Let α_i be the angles at P of corresponding triangles. Then

$$\sum_i \beta_i \geq \sum_i \alpha_i$$

and the equality holds iff $\mathcal{P}$ is planar and convex points P lies in its interior.

Proof.

$$\alpha_i + \gamma_i + \delta_i = \pi$$
$$\delta_{i-1} + \gamma_i + \beta_i \geq \pi$$

Now, sum over all i to get

$$\sum_i \pi \leq \sum_i (\delta_{i-1} + \gamma_i + \beta_i) = \sum_i (\delta_i + \gamma_i + \beta_i) = \sum_i (\pi - \alpha_i) + \beta_i = \sum_i \pi - \sum_i \alpha_i + \sum_i \beta_i$$

Thus

$$\begin{aligned} \sum_i \pi &\leq \sum_i \pi - \sum_i \alpha_i + \sum_i \beta_i \\ 0 &\leq -\sum_i \alpha_i + \sum_i \beta_i \\ \sum_i \alpha_i &\leq \sum_i \beta_i \end{aligned}$$

and $\sum \alpha_i = \sum \beta_i$ iff $\delta_{i-1} + \gamma_i + \beta_i = \pi$ for all i, ie $\mathcal{P}$ is planar. □

Corollary 8.4.

$$\sum_i \beta_i \geq 2\pi$$

Proof. If $\mathcal{P}$ is planar, then $\sum_i \alpha_i = 2\pi = \sum_i \beta_i$. If $\mathcal{P}$ is not planar, choose P in the complex hull of the polygon. Then $2\pi < \sum \alpha_i < \sum \beta_i$. □

Proposition 8.5. The discrete Willmore energy is non-negative, $W(v) \geq 0$ and vanishes iff all vertices of $S(v)$ lie on a sphere and $S(v)$ is convex.

Theorem 8.6. Let S be a compact simplicial surface without boundary. Then $W(S) \geq 0$ and equality hold iff S is a convex polyhedron inscribed in a sphere.

Proposition 8.7. The external angle β between the circumcircles of the triangles is given by any of the equivalent formulas.

$$\begin{aligned} \cos\beta &= -\frac{\text{Re } q}{|q|} = -\frac{\text{Re } (abcd)}{|a||b||c||d|} \\ &= \frac{\langle a, c\rangle\langle b, d\rangle - \langle a, b\rangle\langle c, d\rangle - \langle a, d\rangle\langle b, c\rangle}{|a||b||c||d|} \end{aligned}$$

where $q := q(x_1, x_2, x_3, x_4) \in \mathbb{H}, x_i \in \text{Im } \mathbb{H}, a = x_1 - x_2, b = x_2 - x_3, c = x_3 - x_4, d = x_4 - x_1$.

Proof. 4 points always lie on at least one sphere. Use a Möbius transformation to identify this sphere with a plane $\simeq \mathbb{C}$. Then $a, b, c, d \in \mathbb{C}$. Thus $\frac{a}{d}, \frac{c}{b} \in \mathbb{C}$. Then $\arg\frac{a}{d} = \pi - \alpha, \arg\frac{c}{b} = \pi - \gamma$.

$$\begin{aligned} q = q(x_1, x_2, x_3, x_4) &= \frac{(x_1 - x_2)(x_3 - x_4)}{(x_2 - x_3)(x_4 - x_1)} = \frac{a}{b}\frac{c}{d} = r_1 e^{i(\pi-\alpha)} r_2 e^{i(\pi-\gamma)} \\ &= r_1 r_2 e^{i(2\pi-\alpha-\gamma)} = r e^{i2\pi} e^{i(-\alpha-\gamma)} = r e^{i(-\alpha-\gamma)} \end{aligned}$$

Then $\arg q = -\alpha - \gamma = \beta - \pi$.

$$\frac{Re(q)}{|q|} = \frac{r\cos(\beta - \pi)}{r} = \cos(\beta - \pi) = \cos(\pi - \beta) = -\cos\beta$$

for the first equality. For the second equality recall these identities about complex numbers:

- $|z_1 z_2| = |z_1||z_2|$
- $|\bar{z}| = |z|$
- $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$
- $|z^{-1}| = \left|\frac{\bar{z}}{|z|^2}\right| = \frac{|\bar{z}|}{|z|^2} = \frac{|z|}{|z|^2} = \frac{1}{|z|} = |z|^{-1}$
- $Re(z^{-1}) = \frac{Re(z)}{|z|^2}$
- $Re(z_1 z_2) = Re(z_1)Re(z_2)$

Back to our problem, we get

$$\begin{aligned}\frac{Re(q)}{|q|} &= \frac{Re(ab^{-1}cd^{-1})}{|ab^{-1}cd^{-1}|} = \frac{Re(abcd)}{|b|^2|d|^2|ab^{-1}cd^{-1}|} \\ &= \frac{Re(abcd)}{|b|^2|d|^2|a||b^{-1}||c||d^{-1}|} = \frac{Re(abcd)}{|b|^2|d|^2|a||b|^{-1}|c||d|^{-1}} = \frac{Re(abcd)}{|a||b||c||d|}\end{aligned}$$

For the third equality recall that for $x, y \in Im\mathbb{H}$, $xy = -\langle x, y\rangle + x \times y$ where $-\langle x, y\rangle = Re(xy)$ and $x \times y = Im(xy)$.

$$Re(abcd) = Re((-\langle a, b\rangle + a \times b)(-\langle c, d\rangle + c \times d)) = \langle a, b\rangle\langle c, d\rangle + Re((a \times b)(c \times d))$$

Also $\langle A, B \times C\rangle = \langle B, C \times A\rangle$ and $A \times (B \times C) = B\langle A, C\rangle - C\langle A, B\rangle$

$$\begin{aligned}&= \langle a, b\rangle\langle c, d\rangle - \langle c, d \times (a \times b)\rangle = \langle a, b\rangle\langle c, d\rangle - \Big\langle c, a\langle d, b\rangle - b\langle d, a\rangle \Big\rangle \\ &= \langle a, b\rangle\langle c, d\rangle - \langle a, c\rangle\langle b, d\rangle + \langle b, c\rangle\langle a, d\rangle\end{aligned}$$

□