**Appendix 4.2: Numerical Calculations for Figures 4.1-4.18**

This appendix provides numerical illustrations of the general linkages between the length and intensity of the working day and the paths of productivity and costs. The theoretical linkages are developed in Sections I-III of Chapter 4, formalized in Tables 4.1-4-3, and displayed in Figures 4.1-4.18.

Glossary

$XR$ = cumulative output

$H$ = cumulative labor-hours

$xr \equiv XR/H$ = average productivity of labor

$h$ = length of the given labor shift

$i$ = intensity of labor, $0 \leq i \leq 1$

$l \equiv H/XR$ = labor coefficient = $1/xr$

$A$ = materials input

$a \equiv A/XR$ = materials coefficient

$MK$ = stock of machines

$mk \equiv MK/XR$ = machine coefficient

$mpl = \Delta XR/\Delta H$ = marginal product of labor

$H_{MK}$ = cumulative machine-hours

$mkn = MK/N$ = machines per worker

$l' \equiv N/XR$ = the employment coefficient

$\mathfrak{d}$ = depreciation rate = $0.05$

$p_{MK}$ = machine price

$p_a$ = materials price

$mkh = H_{MK}/H$ = machine-hrs per wrkr-hr

$xr' = XR/N$ = output per worker

$\overline{mkh} = MK/H$ = machines per worker-hour

$w_h$ = hourly wage rate

$w_N$ = wage per worker

$p$ = output price

$afc = \mathfrak{d} \cdot p_{MK} \cdot MK/XR = \mathfrak{d} \cdot p_{MK} \cdot mk$ = avg. fixed cost

$ulc' = w_N \cdot l'$ = unit labor cost with wages paid per worker

$ulc = w_h \cdot l$ = unit labor costs with wages paid per hour

$p_a \cdot a$ = average materials cost

$avc = ulc +$ average materials cost

$ac = afc + avc$ = average (total) cost

$tc$ = total cost = $ac \cdot XR$ for $XR > 0$ and $\mathfrak{d} \cdot p_{MK} \cdot MK$ (fixed cost alone) when $XR = 0$

$mc = \Delta tc/\Delta XR$ = marginal cost

1. Length, Intensity and Productivity of Labor (Figures 4.1-4.4)

The first order of business (so to speak) is to examine the production possibilities of a given technology. The ultimate limit of daily production time is the maximum time that a machine can be operated, which for the sake of illustration is taken to be 20 hours per day. Without labor time, there is no material input and no output, so the starting point is always zero input and zero output. Even in this state, there is a fixed cost corresponding to the depreciation of fixed capital. The productivity of labor is modeled in keeping with the empirical evidence in Section IV of Chapter 4. It is assumed to rise at a slowing rate, peaking at some point below the machine limit of 20 hours. A quadratic function was used to represent the relation between labor productivity and hours worked, in order to allow for a decline in productivity after some labor exhaustion point (12 hours of work in this illustration). To keep matters simple, the intensity of work was

treated as a shift parameter for the productivity curve. For XR= cumulative daily output, H = cumulative daily labor-hours, h = the length of the working day of labor $(1 \leq h \leq 20)$, $i$ = the intensity of labor $(0 \leq i \leq 1)$, and parameters $a_1, a_2, a_3 > 0$, the average hourly productivity of labor xr = XR/H is given by

$$(0.1) \quad xr = (a_1 + a_2 h - a_3 h^2) i$$

The slope of this function at any given intensity is $\partial xr / \partial h = h(a_2 - 2 a_3 \cdot h)i$, which becomes zero at the overwork point $h^* = \frac{a2}{2\,a_3}$. After this point, the productivity of labor decreases with hours worked. In the case of a single shift, the cumulative length of the working day is the same as the shift length, so H = h. The productivity relation implies that total output XR = $h(a_1 + a_2 \cdot h - a_3 \cdot h^2)i$. The output curve turns down at $h^{**} = \frac{-2a_2 - \sqrt{(2a_2)^2 + 12a_1 a_3}}{-6a_3}$, i.e. the productivity of labor becomes negative after this point due to the absolute overextension of the working day. For $0 < h \leq 20$ and parameters $a_1 = 2, a_2 = 1.2, a_3 = 0.05$, the overwork (exhaustion of labor) point is $h^* = 12$ and the point of absolute overextension is $h^{**} = 17$. Finally, raising the intensity of labor raises the productivity of labor at any given number of hours of work, thereby shifting the productivity and output curves upward.

The labor coefficient$(l = H/XR)$ is the inverse of the productivity of labor. For materials A and a given stock of machinesMK = 14, $a \equiv A/XR$ = materials coefficient = 0.3 (assumed to be constant), $mk \equiv \frac{MK}{XR}$ = the machine coefficient (which varies inversely with output) and mpl = $\Delta XR/\Delta H$ = marginal product of labor, we get the following data points and curves. It will be noted that the labor and machine coefficient curves reverse themselves at some point because output actually drops when the working day is extended too far (over-exhaustion leads to damage and destruction). Table 1 displays the numerical values associated with the maximum level of intensity $= 1$. Values for other intensity levels are not shown because they can easily be derived per equation (0.1). Figures 4.1-4.4 in the text depict the main variables for intensities $i = 0, 0.2, 0.4, 0.6, 1$.

[INSERT Appendix 4.2, Table 1: Productivity, Output and Production Coefficients for intensity $i = 1$]

2. Shift Combinations and the Production Possibilities Frontier (Figures 4.5-4.6)

By definition, the neoclassical production possibilities curve is the frontier curve of output as the variable input (labor hours) is changed. This would seem to be the non-decreasing portion of

maximum-intensity output curve in Figure 4.2 of the text (i.e. the curve up to the point at which output peaks). Since machines could be used for a full 20 hours, the firm could increase total daily output by truncating the first shift at 17 hours and adding a second 3-hour shift to yield a new monotonic frontier curve. But it turns out that the resulting $17\colon 3$ curve is not a frontier curve. Indeed, there is no combination that dominates throughout, so that the output frontier consists of changing shift-combinations. Even the shift-combination which produces the highest cumulative daily output $(10\colon 10)$ is below the frontier for the first 10 hours of labor input and does not possess the typical monotonic shape required of a "well-behaved" production function (Varian, 1993, pp. 307-308).

These matters are illustrated by calculating daily output from various shift combinations operated at maximum intensity. The output corresponding to a single 20 hour shift is the same as calculated above in Appendix 4.2 Table 1. One can then use this reference shift to construct all other shift combinations which total 20 hours (the machine limit) by cumulating the output of successive shifts. Thus, a $17\colon 3$ shift will have the same daily cumulative output as the 20 hour reference shift until the $17^{th}$ hour. But in the $18^{th}$ hour, the additional output will come from the first hour of the second shift, and will therefore be the equal in magnitude to the output of the first hour of the reference shift. This same procedure can evidently be repeated for any shift combination. Appendix 4.2 Table 2 maps out these quantities (with the switch points of shifts being highlighted) for shift combinations $20\colon 0$ (the reference shift), $17\colon 3, 3\colon 17, 12\colon 8$ and $10\colon 10$ (the combination which yields maximum output at the end of 20 hours). Figure 4.5 is based on this data. The average and marginal products of labor for the $10\colon 10$ shift are also calculated and used to derive Figure 4.6.

[INSERT Appendix 4.2, Table 2: Production Possibilities Frontier and Shift Combinations at Maximum Intensity ($i = 1$)]

3. Production Function Graphs (Data for Figures 4.7-4.8)

The neoclassical production function charts are derived from a Cobb-Douglas function $YR = MK^{\alpha} \cdot H^{1-\alpha}$, where YR = real net output = real gross output – intermediate input = XR - aXR. Gross output was derived as $XR \equiv YR/(1\text{-}a)$ and plotted against labor hours in Figure 4.7 while gross output per labor hour xr = XR/H was plotted against the machine-labor hour ratio $\overline{mkh} = MK/H$ in Figure 4.8, for a machine stock MK = 14, the materials coefficient a = 0.30 and production function parameter $\alpha = 0.50$.

4. Production patterns for Socially Normal Shift Lengths and Intensity (Figures 4.9-4.14)

This section traces the production patterns arising from the full (20-hour) utilization of a reference plant and equipment operated at normal intensity for two-and-a-half standard (8-hour) labor shifts, as previously developed in text Tables 4.1-4.2. The machine stock $MK = 14$ and the materials coefficient $a = 0.30$ are the same as in all previous examples, and each machine is assumed to require a fixed complement of workers ($N = 1$). Thus, the cumulative daily hours of labor time (H) is the cumulative time spent on successive shifts: from 1 to 8 hours on the first shift, from 9 to 16 hours on the second shift, and from 17 to 20 hours on the final shift. Cumulative machine-hours $H_{MK} = H \cdot MK$ therefore vary from 14 to 280. Finally, the cumulative daily output of these successive shifts is $XR = xr \cdot H$ where xr is the labor productivity corresponding to equation (0.1) in this appendix calculated for normal intensity ($i = 0.80$). These basic variables are then used to calculate $xr' = XR/N = $ output per worker, $mkn = MK/N = $ machines per worker, $xr = XR/H = $ output per worker-hour, $\overline{mkh} = MK/H$ = machines per worker-hour, $mkh = H_{MK}/H = H \cdot MK/H \cdot N = MK/N = $ machine-hours per worker-hour, $mk \equiv \frac{MK}{XR} = $ the machine coefficient, $l' \equiv N/XR = $ the employment coefficient and $l = H/XR = 1/xr = $ the labor coefficient. Appendix 4.2 Table 3 lists the data, highlighting the start of each shift.

[INSERT Appendix 4.2 Table 3: Production patterns for Socially Normal Shift Lengths and Intensity ($i = 0.80$)]

5. Cost Curves and Profits (Figures 4.16-4.18)

The preceding table allows us to derive cost curves and profit for two separate cases: when wages are paid per worker, and when they are paid per hour. In addition to the previously used values of MK and a, $ð = $ the depreciation rate $= 0.05$, $p_{MK} = $ machine price $= 100$, $p_a = $ materials price $= 10$, $w_N = $ wage per worker $= 100$, $w_h = $ hourly wage rate $= 12.5$, and $p = $ output price $= 7$. Output (XR) in Table 4 is taken from Table 3 above. Fixed cost $= $ the amortization on the stock of machines $= ð \cdot p_{MK} \cdot MK = 70$, which yields average fixed cost $afc = ð \cdot p_{MK} \cdot MK/XR = ð \cdot p_{MK} \cdot mk$ for any positive output (i.e. once the first shift starts), where $mk = $ the machine coefficient from Table 3 above. The subsequent calculations come in two sets. In the case of wages paid per worker, unit labor cost $ulc' = w_N \cdot N/XR = w_N \cdot l'$ where $l' = $ the employment coefficient in the previous table. Average variable costs $avc' = ulc' + p_a a$, where the latter term is the unit materials cost, while average (total) cost $ac' = afc' + avc'$. Total cost $tc'$ equals fixed cost alone when there is no output (the shaded entry), but otherwise can be derived as $tc' = ac' XR$. Marginal cost $mc' = \Delta tc'/\Delta XR$. In the case of wages paid per hour, $ulc = w_h \cdot H/XR = w_h l$, where $l = $ the labor coefficient previously derived in Table 3. The calculation

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steps for the other corresponding costs are the same as those in previous case. Finally, total profit for each type of wage payment is derived as total revenue (p·XR) minus the corresponding total costs ($tc'$ or $tc$). Table 4 summarizes this data from which Figures 4.16-4.18 are derived.


[INSERT Appendix 4.2 Table 4: Cost Curves and Profit]