Study Material

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U.G, Sem - 2 (Mathematics)
Paper - CC_3
Integral Calculus

Integration of Some Irrational functions:-

Integration of the expression of the form $\frac{1}{X\sqrt{Y}}$

The Integration of the form $\frac{1}{X\sqrt{Y}}$ is performed with help of the following table

X	Y	Substitution
Linear	Linear	$Y = t^2$
Linear	Quadratic	$X = \frac{1}{t}$
Quadratic	Linear	$Y = t^2$
Quadratic	Quadratic	$\frac{Y}{X} = t^2$

Q. Integrate : $\int \frac{dx}{(2+x)\sqrt{1+x}}$

Sol^n :- Let $I = \int \frac{dx}{(2+x)\sqrt{1+x}}$ —— ①

Let $1+x = z^2 \Rightarrow dx = 2z\,dz$

$x = z^2 - 1 \Rightarrow 2+x = 2+z^2-1 = 1+z^2$

$\therefore$ ① $\Rightarrow I = \int \frac{2z\,dz}{(1+z^2)\sqrt{z^2}} = \int \frac{2\cancel{z}\,dz}{(1+z^2)\cancel{z}}$

$= 2\int \frac{dz}{1+z^2} = 2\tan^{-1} z = 2\tan^{-1}(1+x)$ #

Q. Integrate $\int \frac{dx}{(1+x)\sqrt{1-x^2}}$

Let $I = \int \frac{dx}{(1+x)\sqrt{1-x^2}}$ —— ①

put $1+x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\,dt$

$\therefore x = \frac{1}{t} - 1 = \frac{1-t}{t}$

$1 - x^2 = 1 - \frac{(1-t)^2}{t^2} = \frac{t^2-(1-t)^2}{t^2}$

$= \frac{\cancel{t^2} - 1 + 2t - \cancel{t^2}}{t^2} = \frac{2t-1}{t^2}$

$\Rightarrow 1-x^2 = \frac{2t-1}{t^2}$

$$\therefore \text{①} \Rightarrow I = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{2t-1}{t^2}}} = \int \frac{-\frac{1}{\cancel{t^2}}\,dt}{\frac{1}{\cancel{t^2}}\sqrt{2t-1}}$$

$$= -\int \frac{dt}{\sqrt{2t-1}} = -\int (2t-1)^{-\frac{1}{2}}\,dt$$

$$= -\frac{(2t-1)^{\frac{1}{2}}}{2\times\frac{1}{2}} = -\sqrt{2t-1}$$

$$= -\sqrt{2\frac{1}{1+x}-1} = -\sqrt{\frac{2-(1+x)}{1+x}}$$

$$= -\sqrt{\frac{1-x}{1+x}} \quad \#$$

Q. Integrate: $\displaystyle\int \frac{dx}{(x^2+4)\sqrt{x^2+9}}$

Solⁿ:- Let $I = \displaystyle\int \frac{dx}{(x^2+4)\sqrt{x^2+9}}$ — ①

put $\dfrac{x^2+9}{x^2+4} = u^2 \Rightarrow x^2+9 = u^2x^2+4u^2$

$$\Rightarrow x^2 - u^2x^2 = 4u^2-9$$

$$\Rightarrow x^2(1-u^2) = 4u^2-9$$

$$\Rightarrow x^2 = \frac{4u^2-9}{1-u^2} \quad \text{— ②}$$

$$2x\,dx = \{(1-u^2)(8u) - (4u^2-9)(-2u)\}\,du$$

$\because \dfrac{x^2+9}{x^2+4} = u^2$

taking log both sides

$\log \dfrac{x^2+9}{x^2+4} = \log u^2 \Rightarrow \log(x^2+9) - \log(x^2+4) = \log u^2$

$\Rightarrow \left[\dfrac{2x}{x^2+9} - \dfrac{2x}{x^2+4}\right] dx = 2\dfrac{1}{u}\,du$

$\Rightarrow \dfrac{\cancel{2}x(\cancel{x^2}+4-\cancel{x^2}-9)\,dx}{(x^2+4)(x^2+9)} = \cancel{2}\dfrac{\sqrt{x^2+4}}{\sqrt{x^2+9}}\,du$

$\Rightarrow \dfrac{-5x\,dx}{(x^2+4)\sqrt{(x^2+9)}} = \sqrt{x^2+4}\,du$

$\Rightarrow \dfrac{dx}{(x^2+4)\sqrt{x^2+9}} = -\dfrac{1}{5}\dfrac{\sqrt{x^2+4}}{x} = -\dfrac{1}{5}\sqrt{1+\dfrac{4}{x^2}}\,du$ — ③

Now $1+\dfrac{4}{x^2} = 1+4\cdot\dfrac{(1-u^2)}{4u^2-9}$, by ②

$= \dfrac{4u^2-9+4-4u^2}{4u^2-9} = \dfrac{-5}{4u^2-9} = \dfrac{5}{9-4u^2}$

$\therefore$ ③ $\Rightarrow \displaystyle\int \dfrac{dx}{(x^2+4)\sqrt{x^2+9}} = -\dfrac{1}{5}\int\sqrt{\dfrac{5}{9-4u^2}}\,du$

$= -\dfrac{1}{\sqrt{5}}\displaystyle\int\dfrac{du}{2\sqrt{\frac{9}{4}-u^2}} = \dfrac{-1}{2\sqrt{5}}\int\dfrac{du}{\sqrt{(\frac{3}{2})^2-u^2}}$

$= -\dfrac{1}{2\sqrt{5}}\sin^{-1}\dfrac{2u}{3} = -\dfrac{1}{2\sqrt{5}}\sin^{-1}\left(\dfrac{2}{3}\sqrt{\dfrac{x^2+9}{x^2+4}}\right)$

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