Study - Material

U.G. Sem - VI
Mathematics
paper - CC13

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Fourier Transform

Q. Solve the integral equation

$$\int_0^\infty f(\theta)\cos\alpha\theta \, d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2} dt$.

Soln:- We have $\int_0^\infty f(\theta)\cos\alpha\theta \, d\theta = F_c(\alpha)$

$$\therefore F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases} \quad \text{--- (1)}$$

by inversion formula, we have

$$f(\theta) = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha\theta \, d\alpha$$

$$= \frac{2}{\pi}\int_0^1 (1-\alpha)\cos\alpha\theta \, d\alpha$$

$$= \frac{2}{\pi}\left[\left\{(1-\alpha)\frac{\sin\alpha\theta}{\theta}\right\}_0^1 - \int_0^1 (-1)\frac{\sin\alpha\theta}{\theta}\,d\alpha\right]$$

$$= \frac{2}{\pi}\left\{0 + \left[-\frac{\cos\alpha\theta}{\theta^2}\right]_0^1\right\}$$

$$= -\frac{2}{\pi\theta^2}\left[\cos\theta - 1\right]$$

$$\Rightarrow f(\theta) = \frac{2}{\pi\theta^2}(1-\cos\theta)$$

Now $F_c(\alpha) = \int_0^\infty f(\theta)\cos\alpha\theta\,d\theta$

$$\Rightarrow F_c(\alpha) = \int_0^\infty \frac{2(1-\cos\theta)}{\pi\theta^2}\cos\alpha\theta\,d\theta \quad -(2)$$

From (1) & (2), we get

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta}{\theta^2}\cos\alpha\theta = \begin{cases} 1-\alpha, & 0\le\alpha\le 1 \\ 0, & \alpha>1 \end{cases}$$

Now letting $\alpha\to 0$, we get

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta}{\theta^2}\,d\theta = 1$$

$$\Rightarrow \int_0^\infty \frac{2\sin^2\theta/2}{\theta^2}\,d\theta = \frac{\pi}{2}$$

put $\theta = 2t \Rightarrow d\theta = 2dt$

$$\therefore \int_0^\infty \frac{2\sin^2 t}{4t^2} 2dt = \frac{\pi}{2}$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2} \quad \#$$

Convolution:- (i) The convolution of two functions $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$ is defined as

$$f * g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

(2) **Convolution theorem for Fourier transforms:-** The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms i.e.

$$F\{f(x) * g(x)\} = F\{f(x)\} \cdot F\{g(x)\}.$$

Parseval's identity for Fourier transforms:-
If the Fourier transforms of $f(x)$ and $g(x)$ are respectively $F(S)$ and $G(S)$ respectively, then

(i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\bar{G}(s)\,ds = \int_{-\infty}^{\infty} f(x)\,\bar{g}(x)\,dx$

(ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} [F(s)]^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$

Where bar implies the complex conjugate.

Note:- The following are the Parseval's identities for fourier sine and cosine transforms

(i) $\frac{2}{\pi}\int_{0}^{\infty} F_c(s)\,G_c(s)\,ds = \int_{0}^{\infty} f(x)\,g(x)\,dx$

(ii) $\frac{2}{\pi}\int_{0}^{\infty} F_s(s)\,G_s(s)\,ds = \int_{0}^{\infty} f(x)\,g(x)\,dx$

(iii) $\frac{2}{\pi}\int_{0}^{\infty} [F_c(s)]^2\,ds = \int_{0}^{\infty} \{f(x)\}^2\,dx$

(iv) $\frac{2}{\pi}\int_{0}^{\infty} [F_s(s)]^2\,ds = \int_{0}^{\infty} \{f(x)\}^2\,dx$.