Study - Material

Dr. M. A. Khan
Associate Prof.
Deptt. of Mathematics
Tata College, Chaibasa

U. G. Sem - VI
(Mathematics)
paper - CC_{13}

Fourier Transform

Q. Find the Fourier transform of :

$$f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$

Solⁿ:- $F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$= \int_{-\infty}^{-1} f(x) e^{isx} dx + \int_{-1}^{1} f(x) e^{isx} dx + \int_{1}^{\infty} f(x) e^{isx} dx$$

$$= \int_{-\infty}^{-1} 0 \cdot e^{isx} dx + \int_{-1}^{1} (1-x^2) e^{isx} dx + \int_{1}^{\infty} 0 \cdot e^{isx} dx$$

$$= \int_{-1}^{1} (1-x^2) e^{isx} dx$$

$$= \left[(1-x^2)\frac{e^{isx}}{is} + \frac{2}{is}\int x\, e^{isx}\,dx\right]_{-1}^{1}$$

$$= \left\{(1-x^2)\frac{e^{isx}}{is} + \frac{2}{is}\left[x\frac{e^{isx}}{is} - \int \frac{e^{isx}}{is}\,dx\right]\right\}_{-1}^{1}$$

$$= \left[(1-x^2)\frac{e^{isx}}{is} + 2x\frac{e^{isx}}{(is)^2} - 2\frac{e^{isx}}{(is)^3}\right]_{-1}^{1}$$

$$= \left[\left(0 + \frac{2e^{is}}{-s^2} - \frac{2e^{is}}{(-is^3)}\right) - \left(0 - \frac{2e^{-is}}{(-s^2)} - \frac{2e^{-is}}{(-is^3)}\right)\right]$$

$$= -\frac{2\times 2}{s^2}\left(\frac{e^{is}+e^{-is}}{2}\right) + \frac{2\times 2}{s^3}\left(\frac{e^{is}-e^{-is}}{2i}\right)$$

$$= \frac{-4}{s^2}\cos s + \frac{4}{s^3}\sin s$$

$$\Rightarrow F(s) = -\frac{4}{s^3}(s\cos s - \sin s).$$

Now by inverse fourier transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{4}{s^3}(s\cos s - \sin s)\, ds$$

$$= \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

$$\Rightarrow -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{s\cos s-\sin s}{s^3}e^{-isx}\,ds$$

$$= \begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$$

put $x=\frac{1}{2}$, we obtain

$$\Rightarrow -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{s\cos s-\sin s}{s^3}e^{-\frac{is}{2}}\,ds = 1-\frac{1}{4}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{s\cos s-\sin s}{s^3}\left(\cos\frac{s}{2}-i\sin\frac{s}{2}\right)ds = \frac{3}{4}\times\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{s\cos s-\sin s}{s^3}\left(\cos\frac{s}{2}-i\sin\frac{s}{2}\right)ds = -\frac{3\pi}{8}+0i$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{s\cos s-\sin s}{s^3}\cos\frac{s}{2}\,ds = -\frac{3\pi}{8}$$

$$\Rightarrow 2\int_{0}^{\infty}\frac{s\cos s-\sin s}{s^3}\cos\frac{s}{2}\,ds = -\frac{3\pi}{8}$$

as integral is even function

$$\Rightarrow \int_{0}^{\infty}\frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}$$

Q. Find the Fourier sine transform of $e^{-|x|}$.
Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi e^{-m}}{2}$, $m>0$.

Sol^n:- x being positive in the interval $(0,\infty)$

$\Rightarrow e^{-|x|} = e^{-x}$

$\therefore$ Fourier sine transform of $f(x) = e^{-|x|}$ is given by

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin sx\,dx = \int_0^\infty e^{-x}\sin sx\,dx$$

$$= \left[\frac{e^{-x}}{1+s^2}(-\sin sx - s\cos sx)\right]_0^\infty = \frac{s}{1+s^2}$$

Using inversion formula for fourier sine transform we get

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s\{f(x)\}\sin sx\,ds = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\sin sx\,ds$$

$$\Rightarrow e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\sin sx\,ds$$

$$\Rightarrow e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\sin sm\,ds$$

$$\Rightarrow \int_0^\infty \frac{x}{1+x^2}\sin xm\,dx = \frac{\pi e^{-m}}{2},\ m>0.$$