Study Material

U.G. Sem - V

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(Mathematics)

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paper — CCMATH 511 (CC_{11})

(Dynamics)

Central Orbits

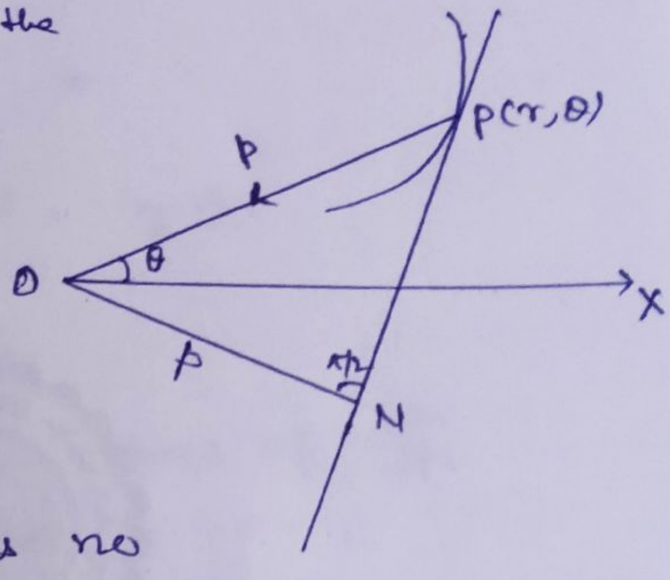
Defn:- Central force :- A force is called a central force if it is always directed towards a fixed point O. This fixed point O is called the centre of force.

Central Orbit :- If a particle moves under the action of central force, the path described by the particle is called central orbit.

Q. A particle moves in a plane curve with an acceleration P which is always directed to a fixed point O in the plane, drive the differential equation of its path.

Soln:- I Differential equation of Central orbit in polar form.

Let the fixed point O be the pole, OX the initial line and the particle at time 't' be in the position P(r, θ).



Clearly the acceleration is radial acceleration and therefore, there is no transversal component of acceleration.

Hence we have

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P \quad \text{—(1)}$$

and $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$ —(2)

Integrating (2)

$$\Rightarrow \quad r^2\frac{d\theta}{dt} = h \text{ (say)} \quad \text{—(3)}$$

Where h is constant.

Let $r = \frac{1}{u}$

diff. w.r.to t

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$$

$$\Rightarrow \frac{dr}{dt} = -r^2\frac{d\theta}{dt}\cdot\frac{du}{d\theta}$$

$$\Rightarrow \frac{dr}{dt} = -h\frac{du}{d\theta}, \text{ by (3)}$$

Again diff. w.r.t t

$$\Rightarrow \frac{d^2r}{dt^2} = -h \frac{d^2u}{d\theta^2} \cdot \frac{d\theta}{dt}$$

$$\Rightarrow \frac{d^2r}{dt^2} = -h \frac{d^2u}{d\theta^2} \cdot hu^2, \text{ by } (3)$$

$$\Rightarrow \frac{d^2r}{dt^2} = -h^2u^2 \frac{d^2u}{d\theta^2}$$

Now substituting the values of $\frac{d^2r}{dt^2}$ and $\frac{d\theta}{dt}$ in (1)

$$\Rightarrow -h^2u^2 \frac{d^2u}{d\theta^2} - \frac{1}{u} h^2u^4 = -P$$

$$\Rightarrow h^2u^2 \left(\frac{d^2u}{d\theta^2} + u \right) = P \quad \text{— (A)}$$

which is the differential equation of central orbit in polar form.

II **Differential equation of central orbit in polar form:-**

Let the perpendicular from O on the tangent at P be ON and let ON = p then

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \quad \text{— (4)}$$

as $u = \frac{1}{r} \Rightarrow \frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta}$

Diff. (4) w.r.to θ, we have

$$-\frac{2}{p^3}\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}$$

$$\Rightarrow -\frac{1}{p^3}\frac{dp}{d\theta} = \frac{du}{d\theta}\left(\frac{d^2u}{d\theta^2}+u\right)$$

$$\Rightarrow -\frac{1}{p^3}\,dp = \frac{P}{h^2u^2}\,du$$, by (A)

$$\Rightarrow -\frac{1}{p^3}\,dp = \left(\frac{P}{h^2}r^2\right)\left(-\frac{1}{r^2}dr\right)$$, as $du = -\frac{1}{r^2}dr$

$$\Rightarrow \boxed{\frac{h^2}{p^3}\frac{dp}{dr} = P}$$, which is the differential equation of central orbit in pedal form

Note:- (i) The expression $r^2\frac{d\theta}{dt}$ is called moment of momentum.

(ii) In a central orbit $r^2\frac{d\theta}{dt} = h$ (constant) i.e. angular momentum is conserved in a central orbit.

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