Study Material

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U.G. Sem - V
(Mathematics)
paper - CCMATH 511 (C11)
(Dynamics)

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Planetary Motion

Q. A particle moves in a path so that its acceleration is always directed towards a fixed point and is equal to $\mu/(\text{distance})^2$. Show that the path is a conic section and distinguish between three cases that arise.

Solⁿ:- The path is a central orbit, since the acceleration is always directed towards a fixed point

$$\therefore\ P = \frac{\mu}{r^2} \quad \text{—— (1)}$$

The pedal form of differential equation of central orbit is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P \quad \text{— (2)}$$

$\therefore$ from (1) & (2)

$$\Rightarrow \frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2}$$

$$\Rightarrow -\frac{2h^2}{p^3}\,dp = -\frac{2\mu}{r^2}\,dr$$

by Integrating, we get

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + c \quad , \text{ as } h = pv \quad \text{— (3)}$$

We know that pedal equations referred to focus as pole of ellipse, parabola, and hyperbola are

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1, \quad p^2 = ar \quad \text{and} \quad \frac{b^2}{p^2} = \frac{2a}{r} + 1 \quad \text{— (4)}$$

respectively, where $2a$ and $2b$ are lengths of major (or transverse) and minor (or conjugate) axes respectively.

Now following three cases arises:

Case I :- elliptic path

Comparing pedal equation of ellipse from (4) with equation (3), we get

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{c}{-1} \Rightarrow h^2 = \frac{\mu b^2}{a} \text{ and } c = -\frac{\mu}{a}$$

$$\therefore v^2 = \frac{2\mu}{r} - \frac{\mu}{a} = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \text{ by} \quad \text{— (5)}$$

From (5) it follows that for the path to be an ellipse if $v^2 < \frac{2\mu}{r}$

Case - II :- Parabolic Path

Comparing the pedal equation of parabola from 4 with 3, we have

$$\frac{h^2}{1} = \frac{2\mu}{1/a} \text{ and } c=0 \Rightarrow h^2 = 2a\mu,\ c=0$$

$$\therefore \text{ 3} \Rightarrow v^2 = \frac{2\mu}{r} \text{ for parabolic path} \quad \text{— 6}$$

Case - III :- Hyperbolic path

Comparing pedal equation of hyperbola from 4 with 3, we have

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{c}{1} \Rightarrow h^2 = \frac{\mu b^2}{a} \text{ and } c = \frac{\mu}{a}$$

$$\therefore \text{ 3 becomes, } v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right) \quad \text{— 7}$$

From equation 7 it is follow that the path is a hyperbola if $v^2 > \frac{2\mu}{r}$.

Kepler's law for planetary motion

Kepler discovered the following three laws governing the motion of various planets about the Sun.

1. Each planet describes an ellipse with the Sun is one of its foci.

(2) The area desribed by the radii drawn from the planet to Sun are in the same orbit proportional to the times of describing them.

(3) The squares of the periodic times of the planets are proportional to the cubes of the major axes of their orbits.

Q. A particle describes an ellipse under a force $\frac{\mu}{(\text{distance})^2}$ towards the focus. If it was projected with velocity V from a point distant r from the centre of force, show that its periodic time is $\frac{2\pi}{\sqrt{\mu}}\left[\frac{2}{r}-\frac{V^2}{\mu}\right]^{-3/2}$.

Soln:- Here $V^2 = \mu\left(\frac{2}{r}-\frac{1}{a}\right)$

$$\Rightarrow \frac{V^2}{\mu} = \frac{2}{r} - \frac{1}{a}$$

$$\Rightarrow \frac{1}{a} = \frac{2}{r} - \frac{V^2}{\mu} \quad \text{—— (1)}$$

$\therefore$ Periodic time T is given by

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

$$\Rightarrow T = \frac{2\pi}{\mu}\left[\frac{2}{r} - \frac{V^2}{\mu}\right]^{-3/2}$$

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