Study Material

U.G. Sem – VI (Mathematics)

paper – CC14

Laplace Transform

Page – ①
Date: 13/5/20
Dr. M. A. Khan
Associate Prof.
Deptt. of Mathematics
Tata College, Chaibasa

Change of scale property :-

Theorem:- If $L\{f(t)\} = F(p)$, then $L\{f(at)\} = \frac{1}{a}F\left(\frac{p}{a}\right)$

Proof:- By definition, we have

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\,dt$$

$$\therefore L\{f(at)\} = \int_0^\infty e^{-pt} f(at)\,dt \quad \text{—— ①}$$

put $at = z$ so that $a\,dt = dz$.

$$\therefore \text{R.H.S of ①} = \int_0^\infty e^{-p\left(\frac{z}{a}\right)} f(z)\,\frac{dz}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{p}{a}\right)z} f(z)\,dz$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{p}{a}\right)t} f(t)\,dt$$, by changing the variable of integration

$$= \frac{1}{a}F\left(\frac{p}{a}\right)$$, as $F(p) = \int_0^\infty e^{-pt} f(t)\,dt$

$$\Rightarrow L\{f(at)\} = \frac{1}{a}F\left(\frac{p}{a}\right) \quad \#$$

2020.05

Theorem:- If $L\{f(t)\} = F(p)$ then
$$L\{t\,f(t)\} = -F'(p) = (-1)\frac{d}{dp}F(p)$$

Proof:- We have $F(p) = L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\,dt$

$$\Rightarrow F'(p) = \frac{d}{dp}\int_0^\infty e^{-pt} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial p}\{e^{-pt} f(t)\}\,dt,$$ by Leibnitz rule for differentiating under the sign of integration

$$= \int_0^\infty -t\,e^{-pt} f(t)\,dt$$

$$= -\int_0^\infty e^{-pt}\{t\,f(t)\}\,dt$$

$$= -L\{t\,f(t)\}$$

$$\Rightarrow L\{t\,f(t)\} = -F'(p). \#.$$

Theorem:- If $L\{f(t)\} = F(p)$ then
$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{dp^n}F(p), \; n = 1, 2, 3, \ldots$$

Proof:- We shall prove the theorem by mathematical Induction.

We have prove that
$L\{t\,f(t)\} = (-1)\frac{d}{dp}F(p)$ i.e. Theorem is true for $n = 1$.

2020.05.13

Let us suppose that theorem is true for $n=r$

i.e $L\{t^r f(t)\} = (-1)^r \frac{d^r F(p)}{dp^r}$ — ①

$$\Rightarrow \int_0^\infty e^{-pt} t^r f(t)\,dt = (-1)^r \frac{d^r}{dp^r} F(p) \quad — ②$$

Now differentiating both sides of ② w.r.t p

$$\Rightarrow \frac{d}{dp}\int_0^\infty e^{-pt} t^n f(t)\,dt = (-1)^r \frac{d^{r+1}}{dp^{r+1}} F(p) \quad — ③$$

Now using the Leibnitz rule for differentiating under the sign of integral L.H.S of ③ becomes

$$\int_0^\infty \frac{\partial}{\partial p}\{e^{-pt} t^n f(t)\,dt\} = \int_0^\infty e^{-pt}(-t)\, t^r f(t)\,dt$$

$$= -\int_0^\infty e^{-pt} t^{r+1} f(t)\,dt$$

Hence from ③

$$-\int_0^\infty e^{-pt} t^{r+1} f(t)\,dt = (-1)^r \frac{d^{r+1}}{dp^{r+1}} F(p)$$

$$\Rightarrow \int_0^\infty e^{-pt} t^{r+1} f(t)\,dt = (-1)^{r+1} \frac{d^{r+1}}{dp^{r+1}} F(p)$$

$$\Rightarrow L\{t^{r+1} f(t)\} = (-1)^{r+1} \frac{d^{r+1}}{dp^{r+1}} F(p)$$

i.e true for $n = r+1$

Therefore by mathematical induction the theorem is true for all finite n.

Q. Find (i) $L\{t \cos at\}$ (ii) $L\{t^n e^{at}\}$.

Sol^n:- (i) $L\{t \cos at\}$

$$\because L\{\cos at\} = \frac{p}{p^2+a^2}, \quad p>0$$

$$L\{t \cos at\} = -\frac{d}{dp} F(p) = -\frac{d}{dp} L\{\cos at\}$$

$$= -\frac{d}{dp}\left\{\frac{p}{p^2+a^2}\right\} = -\left[\frac{(p^2+a^2)\cdot 1 - p\cdot 2p}{(p^2+a^2)^2}\right]$$

$$= \frac{p^2-a^2}{(p^2+a^2)^2}$$

(ii) $L\{t^n e^{at}\}$ =

We have $L(e^{at}) = \frac{1}{p-a}, \quad p>a$

$$\therefore L\{t^n e^{at}\} = (-1)^n \frac{d^n}{dp^n}\left\{\frac{1}{(p-a)}\right\}$$

$$= (-1)^n \frac{(-1)^n \lfloor n}{(p-a)^{n+1}} = \frac{\lfloor n}{(p-a)^{n+1}}$$

#