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Study - Material

U.G. Sem - VI (Mathematics)
paper - CC_{13}

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Fourier Transform

Integral Transform:- The integral transform of a function $f(x)$ denoted by $I[f(x)]$, is defined by

$$\bar{f}(s) = \int_{x_1}^{x_2} f(x)\, K(s, x)\, dx$$

Where $K(s, x)$ is called the Kernel of the transform and is a known function of s and x. The function $f(x)$ is called the inverse transform of $\bar{f}(s)$.

When $K(s, x) = e^{isx}$, we have the Fourier transform of $f(x)$, i.e.

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

Note:- $f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$ is called inverse Fourier transform of $F(s)$.

Fourier Sine and Cosine transform:-

Def^n:- (i) Fourier Sine transform the function $f(x)$ is defined as

$$F_s(s) = \int_0^\infty f(x) \sin sx \, dx$$

Note:- $f(x) = \frac{2}{\pi}\int_0^\infty F_s(s) \sin sx \, dx$ is called the inverse Fourier Sine transform of $F_s(s)$.

Def^n:- (ii) Fourier Cosine transform of the function $f(x)$ is defined as

$$F_c(s) = \int_0^\infty f(x) \cos sx \, dx$$

Note:- $f(x) = \frac{2}{\pi}\int_0^\infty F_c(s) \cos sx \, dx$

is called the inverse Fourier Cosine transform of $F_c(s)$.

Q. Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \frac{\sin x}{x} \, dx$

Soln.:- The Fourier transform of $f(x)$ is

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$= \int_{-\infty}^{-1} f(x) e^{isx} \, dx + \int_{-1}^{1} f(x) e^{isx} \, dx + \int_{1}^{\infty} f(x) e^{isx} \, dx$$

$$= \int_{-\infty}^{-1} 0 e^{isx} \, dx + \int_{-1}^{1} 1 \cdot e^{isx} \, dx + \int_{1}^{\infty} 0 \, e^{isx} \, dx$$

$$\Rightarrow F[f(x)] = \int_{-1}^{1} e^{isx} \, dx = \left[\frac{e^{isx}}{is} \right]_{-1}^{1}$$

$$= \frac{e^{is} - e^{-is}}{is} = \frac{2 \sin s}{s}, \quad s \neq 0$$

Now by inverse Fourier transform, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \, e^{-isx} \, ds$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin s}{s}\, e^{-isx}\, ds$$

$$\Rightarrow \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|>0 \end{cases} = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\cancel{2}\sin s}{s}\, e^{-isx}\, ds$$

putting $x = 0$

$$\Rightarrow 1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \pi,$$ as $\frac{\sin s}{s}$ is an even function

$$\Rightarrow \int_{0}^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}$$

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