Study Material

U. G. Sem - V

(Mathematics)

paper - CCMATH 511 (CC11)

(Dynamics)

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Q A particle moves with a central acceleration $\mu\left\{r+\frac{a^4}{r^3}\right\}$ being projected from an apse at a distance a with velocity $2a\sqrt{\mu}$; show that it describes the curve $r^2(2+\cos\sqrt{3}\,\theta)=3a^2$.

Solⁿ:- Here $P=\mu\left(r+\frac{a^4}{r^3}\right)=\mu\left(u^{-1}+a^4u^3\right)$

Differential equation of the central orbit is

$$h^2\left\{\frac{d^2u}{d\theta^2}+u\right\}=\frac{P}{u^2}$$

$$\Rightarrow h^2\left\{\frac{d^2u}{d\theta^2}+u\right\}=\frac{\mu\left(u^{-1}+a^4u^3\right)}{u^2}=\mu\left(u^{-3}+a^4u\right)$$

multiply both sides by $2\frac{du}{d\theta}$ and integrate

$$\Rightarrow v^2=h^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=\mu\left(-u^{-2}+a^4u^2\right)+A \quad \text{--- (1)}$$

Initially at an apse.

$$u=\frac{1}{a},\quad \frac{du}{d\theta}=0,\quad v=2a\sqrt{\mu}$$

$$(1)\Rightarrow 4a^2\mu=h^2\left\{0+\frac{1}{a^2}\right\}=\mu\left(-a^2+a^2\right)+A.$$

$$\Rightarrow h^2=4a^4\mu,\quad A=4a^2\mu$$

$$(1) \Rightarrow 4a^4\mu\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = \mu(-u^{-2} + a^4u^2) + 4a^2\mu$$

$$\Rightarrow 4a^4\left(\frac{du}{d\theta}\right)^2 + 4a^4u^2 = -u^{-2} + a^4u^2 + 4a^2$$

$$\Rightarrow 4a^4\left(\frac{du}{d\theta}\right)^2 = -\frac{1}{u^2} + a^4u^2 + 4a^2 - 4a^4u^2$$

$$\Rightarrow 4a^4\left(\frac{du}{d\theta}\right)^2 = -\frac{1}{u^2} + 4a^2 - 3a^4u^2$$

$$\Rightarrow 4a^4u^2\left(\frac{du}{d\theta}\right)^2 = -1 + 4a^2u^2 - 3a^4u^4$$

$$= \frac{1}{3} - \frac{4}{3} + 4a^2u^2 - 3a^4u^4$$

$$= \left(\frac{1}{\sqrt{3}}\right)^2 - \left\{(\sqrt{3}a^2u^2)^2 - 2\sqrt{3}a^2u^2\cdot\frac{2}{\sqrt{3}} + \left(\frac{2}{\sqrt{3}}\right)^2\right\}$$

$$\Rightarrow 4a^4u^2\left(\frac{du}{d\theta}\right)^2 = \left(\frac{1}{\sqrt{3}}\right)^2 - \left\{\sqrt{3}a^2u^2 - \frac{2}{\sqrt{3}}\right\}^2$$

$$\Rightarrow 2a^2u\frac{du}{d\theta} = -\sqrt{\left[\left(\frac{1}{\sqrt{3}}\right)^2 - \left\{\sqrt{3}a^2u^2 - \frac{2}{\sqrt{3}}\right\}^2\right]}$$

$$\Rightarrow d\theta = -\frac{2a^2u\,du}{\sqrt{\left[\left(\frac{1}{\sqrt{3}}\right)^2 - \left\{\sqrt{3}a^2u^2 - \frac{2}{\sqrt{3}}\right\}^2\right]}}$$

putting $\sqrt{3}a^2u^2 - \frac{2}{\sqrt{3}} = t$

$$\Rightarrow 2\sqrt{3}a^2u\,du = dt$$

$$\therefore \int d\theta = \frac{1}{\sqrt{3}}\int\frac{-dt}{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 - t^2}}$$

$$\Rightarrow \sqrt{3}\theta = \cos^{-1}(\sqrt{3}t) + B \quad \text{—— (2)}$$

Initially $u = \frac{1}{a}$, $\theta = 0$, $t = \frac{1}{\sqrt{3}}$

$\therefore$ (2) $\Rightarrow$ $0 = \cos^{-1} 1 + B \Rightarrow B = 0$

$\therefore$ (2) $\Rightarrow$ $\sqrt{3}\,\theta = \cos^{-1}(\sqrt{3}\,t)$

$\Rightarrow \sqrt{3}\,t = \cos(\sqrt{3}\,\theta)$

$\Rightarrow \sqrt{3}\left(\sqrt{3}\,a^2u^2 - \frac{2}{\sqrt{3}}\right) = \cos\sqrt{3}\,\theta$

$\Rightarrow 3a^2u^2 = 2 + \cos\sqrt{3}\,\theta$

$\Rightarrow 3a^2 = r^2(2 + \cos\sqrt{3}\,\theta)$

$\Rightarrow \boxed{r^2(2 + \cos\sqrt{3}\,\theta) = 3a^2}$

Q. A particle subjected to an acceleration $\mu(u^4 + 2au^5)$ projected from a point $(a, 0)$ at an angle $\cot^{-1} 2$ with the initial line with a velocity equal to the velocity from infinity; prove that the equation of the path is $r = a(1 + 2\sin\theta)$.

Solⁿ:- Here $P = \mu(u^4 + 2au^5)$

Let V be the velocity from infinity at the same distance under the same acceleration, then

$$\frac{V^2}{2} = -\int_{\infty}^{a} \mu\left(\frac{1}{r^4} + \frac{2a}{r^5}\right) dr = \frac{5\mu}{6a^3}$$

$\therefore V = \sqrt{\frac{5\mu}{3a^3}}$ and this is the velocity of projection

Differential equation of the central orbit is

$$h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\frac{P}{u^2}$$

$$\Rightarrow h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\frac{\mu(u^4+2au^5)}{u^2}=\mu(u^2+2au^3)$$

multiply both sides by $2\frac{du}{d\theta}$ and integrating

$$\Rightarrow v^2=h^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=\mu\left(\frac{2u^3}{3}+au^4\right)+A \quad \text{---(1)}$$

Initially direction of projection makes an angle $\cot^{-1}2$ with the initial line

Hence $\phi=\cot^{-1}2 \Rightarrow 2=\cot\phi \Rightarrow \sin\phi=\frac{1}{\sqrt{5}}$

Let p_0 be the length of perpendicular from the origin on the tangent in the initial position. Also $r=a$

$\therefore p=r\sin\phi \Rightarrow p_0=a.\frac{1}{\sqrt{5}}=\frac{a}{\sqrt{5}}$

Also $\left(\frac{du}{d\theta}\right)^2+u^2=\frac{1}{p^2} \Rightarrow \left(\frac{du}{d\theta}\right)^2+u^2=\frac{1}{p_0^2}=\frac{5}{a^2}$ initially

$$(1)\Rightarrow \frac{5\mu}{3a^3}=\frac{5h^2}{a^2}=\mu\left(\frac{2}{3a^3}+\frac{1}{a^3}\right)+A$$

$$\Rightarrow h^2=\frac{\mu}{3a}, \quad A=0$$

$$\therefore (1)\Rightarrow h^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=\mu\left(\frac{2u^3}{3}+au^4\right)$$

$$\Rightarrow \frac{\mu}{3a}\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=\mu\left(\frac{2u^3}{3}+au^4\right)$$

$\Rightarrow \left(\frac{du}{d\theta}\right)^2 + u^2 = 2au^3 + 3a^2u^4$

$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = 2au^3 + 3a^2u^4 - u^2$

$\Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{2a}{r^3} + \frac{3a^2}{r^4} - \frac{1}{r^2}$

$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = 2ar + 3a^2 - r^2$

$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = 4a^2 - (r^2 - 2ar + a^2)$

$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = 4a^2 - (r-a)^2$

$\Rightarrow \frac{dr}{d\theta} = \sqrt{(2a)^2 - (r-a)^2}$

$\Rightarrow \int d\theta = \int \frac{dr}{\sqrt{(2a)^2 - (r-a)^2}}$

$\Rightarrow \theta = \sin^{-1}\frac{(r-a)}{2a} + B$ — (2)

Initially, $r = a$, $\theta = 0$ {i.e. at the point $(a,0)$}

$\therefore$ $B = 0$

$\therefore$ (2) $\Rightarrow \sin\theta = \frac{(r-a)}{2a} \Rightarrow r - a = 2a\sin\theta$

$\Rightarrow \boxed{r = a(1 + 2\sin\theta)}$ #