Study Materials for
U. G. Sem - III (Mathematics)
paper - CC6
(Group Theory)

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Theorem :- A non-empty subset H of a finite group G is a subgroup of G iff

$$a \in H,\ b \in H \Rightarrow ab \in H.$$

Proof :- Suppose H is a subgroup of G.
$\Rightarrow$ H itself a group
Therefore $a \in H,\ b \in H \Rightarrow ab \in H$.
Conversely suppose that

$$a \in H,\ b \in H \Rightarrow ab \in H$$

Let $a \in H$ be any element. Then by the given condition
$a^2 = a.a \in H$, $a^3 = aa^2 \in H$, $a^4 = aa^3 \in H$ and so on.
Thus proceeding in this way, we get $a^n \in H$ where n is any positive integer
Thus the infinite collection of elements $a, a^2, a^3, \ldots a^n, \ldots$ all belong to H

But H is a finite subset of G.
Therefore there must be repetitions in this collection of elements.
Hence there must exist an identity $e \in H$ such that $a^n = e$ which will be also the identity element of G.

Also $a^{n-1} \in H$

$\Rightarrow a^n a^{-1} \in H$

$\Rightarrow e a^{-1} \in H$

$\Rightarrow a^{-1} \in H$

Thus $a \in H \Rightarrow a^{-1} \in H$

$\Rightarrow$ H is a Subgroup of G.

Theorem :- If H_1 and H_2 be two subgroups of a group G, then $H_1 \cap H_2$ is also a subgroup of G.

Proof :- Let $x_1, x_2 \in H_1 \cap H_2$

$\Rightarrow x_1, x_2 \in H_1$ and $x_1, x_2 \in H_2$

$\because x_1, x_2 \in H_1$ and H_1 is a Subgroup of G

$\Rightarrow x_1 x_2^{-1} \in H_1$

also $x_1, x_2 \in H_2$ and H_2 is a subgroup of G

$\Rightarrow x_1 x_2^{-1} \in H_2$

$\therefore x_1 x_2^{-1} \in H_1$ and $x_1 x_2^{-1} \in H_2$

$\Rightarrow x_1 x_2^{-1} \in H_1 \cap H_2$

$\Rightarrow H_1 \cap H_2$ is a subgroup of G.

Note:- If H_1 and H_2 be two subgroups of G it is not generally true that $H_1 \cup H_2$ is a subgroup of G.

Example:- Let G be the additive group of integers i.e.

$$G = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$$

Let $H_1 = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$

and $H_2 = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$

Then $H_1 \cup H_2 = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \ldots\}$

which is not a group as

$$2 + 3 = 5 \notin H_1 \cup H_2$$

$\Rightarrow H_1 \cup H_2$ is not a subgroup of G. #

Theorem:- For any element a in a group G
(i) $a^m a^n = a^{m+n}$ (ii) $(a^m)^n = a^{mn}$.
for all integral values of m and n.

Proof:- (i) Case — I
Let m and n be positive integers.

$\Rightarrow a^m a^n = (a.a.a\ldots \text{to } m \text{ factors})(a.a.a\ldots n \text{ factors})$
$= a.a.a\ldots \text{to } (m+n) \text{ factor}$
$= a^{m+n}$

$\Rightarrow a^m a^n = a^{m+n}$

Case — II Let m and n both negative integers.
Let $m = -r$, $n = -s$ where r and s are positive integers

$\Rightarrow a^m a^n = a^{-r} a^{-s}$
$= (a^{-1})^r (a^{-1})^s$
$= (a^{-1})^{r+s}$
$= a^{-r-s}$

$\Rightarrow a^m a^n = a^{m+n}$

Case - III: when any one is positive and the other is negative
Let m be a positive and n be a negative integer, say $n = -p$, where p is a positive integer

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$$\Rightarrow a^m a^n = a^m a^{-p} = a^m (a^{-1})^p$$

$$= (a.a.a. \cdots \text{to } m \text{ factors})(a^{-1}.a^{-1}.a^{-1} \cdots \text{to } p \text{ factors})$$

$$= \begin{cases} a^{m-p}, & \text{if } m \geq p \\ (a^{-1})^{p-m}, & \text{if } p \geq m \end{cases}$$

$$= a^{m-p} = a^{m+n}.$$

ii) Case I: Let n be a positive integer

$$\Rightarrow (a^m)^n = a^m.a^m.a^m \cdots \text{to } n \text{ factors}$$

$$= a^{m+m+m+\cdots} \text{ to } n \text{ factors}$$

$$= a^{mn}$$

Case II: Let n be a negative integer, say $n = -p$ where p is a positive integer.

$$\Rightarrow (a^m)^n = (a^m)^{-p} = \{(a^m)^{-1}\}^p$$

$$= (a^{-m})^p = a^{-mp}$$

$$= a^{mn}$$

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