Study Materials for

U.G. Sem - III (Mathematics)

paper - CC6

(Group Theory)

Date: 6/5/20

Dr. M. A. Khan
Associate Professor
Deptt. of Mathematics
Tata College,
Chaibasa

## Cancellation Law in a group.

**Theorem:-** If $a, b, c \in G$, then

(i) $ab = ac \Rightarrow b = c$ (Left Cancellation law)

(ii) $ba = ca \Rightarrow b = c$ (Right Cancellation Law)

**Proof:-** (i) Given that $ab = ac$

$\because$ $a \in G$ and $G$ is a group, then $a^{-1} \in G$.

$\because$ $ab = ac$

$\Rightarrow a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$, by associative law

$\Rightarrow eb = ec$, by inverse law

$\Rightarrow b = c$, by identity law.

(ii) given that

$ba = ca$

$\Rightarrow (ba)a^{-1} = (ca)a^{-1}$

$\Rightarrow b(aa^{-1}) = c(aa^{-1})$, by associative law

$\Rightarrow be = ce$, by inverse law

$\Rightarrow b = c$, by identity law.

**Theorem:-** In a group $G$, prove that $(ab)^{-1} = b^{-1}a^{-1}$, where $a, b \in G$.

**Proof:-** $\because a, b \in G$

$\Rightarrow a^{-1}, b^{-1} \in G$

Now $(b^{-1}a^{-1})(ab) = b^{-1}\{(a^{-1}a)b\}$, by associative law

$= b^{-1}(eb)$, by inverse law

$= b^{-1}b$, by identity law

$= e$, by inverse law

$\Rightarrow (b^{-1}a^{-1})(ab) = e$ —— (1)

also $(ab)(b^{-1}a^{-1})$

$= a\{(bb^{-1})a^{-1}\}$, by associative law

$= a(ea^{-1})$, by inverse law

$= aa^{-1}$, by identity law

$= e$, by inverse law

$\Rightarrow (ab)(b^{-1}a^{-1}) = e$ —— (2)

2020.05.0

From ① & ②

$(b^{-1}a^{-1})(ab) = e = (ab)(b^{-1}a^{-1})$

$\Rightarrow (ab)^{-1} = b^{-1}a^{-1}$. #

Q. Prove that $(ab)^2 = a^2b^2$ for all choices of $a, b \in G$ iff $G$ is abelian.

Proof :- Suppose $G$ is abelian group.

$\Rightarrow ab = ba$

$\therefore (ab)^2 = (ab)(ab)$

$= a(ba)b$, as $G$ is associative

$= a(ab)b$, as $G$ is abelian

$= (aa)(bb)$, as $G$ is associative

$= a^2b^2$

$\Rightarrow (ab)^2 = a^2b^2$

Conversely suppose that

$(ab)^2 = a^2b^2$

$\Rightarrow (ab)(ab) = (aa)(bb)$

$\Rightarrow a(ba)b = a(ab)b$, by associative law

$\Rightarrow (ba)b = (ab)a$, by left cancellation law

$\Rightarrow ba = ab$, by right cancellation law

$\Rightarrow G$ is an abelian group.

## Subgroups:-

**Definition**:- Let $G$ be a group and $H$ a subset of $G$. Then $H$ is said to be a subgroup of $G$ if $H$ is a group under the group operation of $G$.

Ex:-1. The set of integers $I$ is an additive group. The set $E$ of even integers is a subset of $I$ and is a group under addition. Hence $E$ is a subgroup of $I$.

Ex:-2. The set $R^*$ of non-zero real numbers is a group under multiplication.
The set $Q^*$ of non-zero rational numbers is a subset of $R^*$ and is a group under multiplication
Hence $Q^*$ is a subgroup of $R^*$.

**Theorem**:- Let $G$ be a group and $H$ a non-empty subset of $G$. Then $H$ is a subgroup of $G$ if and only if

$$a \in H,\ b \in H \Rightarrow ab^{-1} \in H.$$

**Proof**:- Suppose that
for $a \in H, b \in H \Rightarrow ab^{-1} \in H$. —(1)

$\because a \in H \Rightarrow a \in G$

$\Rightarrow$ H is a subset of G.

$\because a \in H, a \in H \Rightarrow aa^{-1} \in H$

$\Rightarrow e \in H$

$\Rightarrow$ e is an Identity of H.

$\because e \in H$ and $a \in H$

$\Rightarrow ea^{-1} \in H$

$\Rightarrow a^{-1} \in H$

Thus $a \in H \Rightarrow a^{-1} \in H$

Now $a \in H, b^{-1} \in H \Rightarrow a(b^{-1})^{-1} \in H$

$\Rightarrow ab \in H$

$\Rightarrow$ H satisfy the closure property

The binary operation in G is associative and H is a subset of G, it must be associative in H.

$\Rightarrow$ H forms a group. $\Rightarrow$ H is a subgroup of G.

Conversely suppose that H is a subgroup of G.

$\because$ H is subgroup of G

$\Rightarrow$ H is a group itself

$\therefore b \in H \Rightarrow b^{-1} \in H$

By closure property $a \in H, b^{-1} \in H \Rightarrow ab^{-1} \in H$

#