Study Material

U. G. Sem – VI (Mathematics)
Paper — CC_{14}

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Laplace Transform :-

Def^n:- Let $f(t)$ be a real function defined for all real $t > 0$

Define $F(p) = \int_0^\infty e^{-pt} f(t)\, dt$

for all value of p for which this improper integral exists. The function F is called the Laplace Transform of f and we write

$L\{f\} = F$ or $L\{f(t)\} = F(p)$

Q. Find the Laplace transform of

(i) $f(t) = c$, where $c \neq 0$ is a constant

(ii) $f(t) = t$ (iii) $f(t) = e^{at}$, $p > a$

Sol^n:- By def^n

$$L(c) = \int_0^\infty e^{-pt} c\, dt = c\int_0^\infty e^{-pt}$$

$$= c\left[\frac{e^{-pt}}{-p}\right]_0^\infty = c\left[0 - \left(-\frac{1}{p}\right)\right] = \frac{c}{p}$$

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Note :- in particular, if $c=1$, then $L\{1\} = \frac{1}{p}$

(ii) $L(t) = \int_0^\infty e^{-pt} \cdot t \, dt$

$= \left[t \frac{e^{-pt}}{p} \right]_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-pt}}{-p} dt$

$= [0-0] + \frac{1}{p} \int_0^\infty e^{-pt} dt$

$= \frac{1}{p} \left[\frac{e^{-pt}}{-p} \right]_0^\infty = \frac{1}{p} \cdot \frac{1}{p} = \frac{1}{p^2}$

$\Rightarrow L(t) = \frac{1}{p^2}$

(iii) $L\{e^{at}\} = \int_0^\infty e^{-pt} e^{at} dt$

$= \int_0^\infty e^{-(p-a)t} dt = \left[\frac{e^{-(p-a)t}}{-(p-a)} \right]_0^\infty$

$= \frac{1}{p-a}$

$\Rightarrow L\{e^{at}\} = \frac{1}{(p-a)}$

Q. Find (i) $L\{t^n\}$, if n is a positive integer

(ii) $L\{\sin at\}$ (iii) $L\{\cos at\}$

Sol^n: $\because L\{t^n\} = \int_0^\infty e^{-pt} t^n dt$

$= \left[\frac{t^n e^{-pt}}{-p} \right]_0^\infty + \frac{n}{p} \int_0^\infty t^{n-1} e^{-pt} dt$

$$= [0-0] + \frac{n}{p}\int_0^\infty e^{-pt} t^{n-1}\,dt$$

$$\Rightarrow L\{t^n\} = \frac{n}{p}\int_0^\infty e^{-pt} t^{n-1}\,dt$$

Continuing this process, we get

$$L\{t^n\} = \frac{n(n-1)(n-2)\cdots 3.2.1}{p.p.p\ldots p\,(n\text{-times})}\int_0^\infty e^{-pt}\,dt$$

$$= \frac{\lfloor n}{p^n}\left[\frac{e^{-pt}}{-p}\right]_0^\infty = \frac{\lfloor n}{p^{n+1}} = \frac{\Gamma(n+1)}{p^{n+1}},\ p>0$$

(ii) $L\{\sin at\} = \int_0^\infty e^{-pt}\sin at\,dt$

$$= \left[\frac{e^{-pt}}{p^2+a^2}\{-p\sin at + a\cos at\}\right]_0^\infty$$

$$= \frac{a}{p^2+a^2},\ p>0$$

Similarly $L\{\cos at\} = \int_0^\infty e^{-pt}\cos at = \frac{p}{p^2+a^2}, p>0$

Q. Find (i) $L\{\cosh at\}$ (ii) $L\{\sinh at\}$

Solⁿ:- $L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$$= \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{p-a} + \frac{1}{p+a}\right] = \frac{1}{2}\,\frac{p+\cancel{a}+p-\cancel{a}}{(p-a)(p+a)} = \frac{p}{p^2-a^2}$$

(iv) $L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$

$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$

$= \frac{1}{2}\left[\frac{1}{p-a} - \frac{1}{p+a}\right] = \frac{1}{2}\left[\frac{(\cancel{p}+a) - (\cancel{p}-a)}{(p-a)(p+a)}\right]$

$\Rightarrow L\{\sinh at\} = \frac{a}{p^2 - a^2}$

Q. find the Laplace transform of the function $f(t) = (\sin t - \cos t)^2$

Soln:- we have $f(t) = (\sin t - \cos t)^2$

$\Rightarrow f(t) = \sin^2 t + \cos^2 t - 2\sin t \cdot \cos t$

$= 1 - \sin 2t$

$\therefore L\{f(t)\} = L\{1 - \sin 2t\}$

$= L\{1\} - L\{\sin 2t\}$

$= \frac{1}{p} - \frac{2}{p^2+4}$ #