Study Material

U.G, Sem - VI (Mathematics)

paper - CG4

Laplace Transform

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Q Find $L\{\sin\sqrt{t}\}$

Solⁿ:- We know that

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots \infty$$

$$\therefore \sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{\lfloor 3} + \frac{t^{5/2}}{\lfloor 5} - \frac{t^{7/2}}{\lfloor 7} + \cdots \infty$$

$$\Rightarrow L\{\sin\sqrt{t}\} = L\left\{t^{1/2} - \frac{t^{3/2}}{\lfloor 3} + \frac{t^{5/2}}{\lfloor 5} - \frac{t^{7/2}}{\lfloor 7} + \cdots \infty\right\}$$

$$= L(t^{1/2}) - \frac{1}{\lfloor 3} L(t^{3/2}) + \frac{1}{\lfloor 5} L(t^{5/2}) - \frac{1}{\lfloor 7} L(t^{7/2}) + \cdots \infty$$

$$= \frac{\Gamma\frac{3}{2}}{p^{3/2}} - \frac{1}{\lfloor 3}\frac{\Gamma\frac{5}{2}}{p^{5/2}} + \frac{1}{\lfloor 5}\frac{\Gamma\frac{7}{2}}{p^{7/2}} - \frac{1}{\lfloor 7}\frac{\Gamma\frac{9}{2}}{p^{9/2}} + \cdots \infty$$

as $L(t^n) = \frac{\Gamma n+1}{p^{n+1}}$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}} - \frac{\frac{1}{2}\cdot\frac{3}{2}\sqrt{\pi}}{\lfloor 3\, p^{5/2}} + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\sqrt{\pi}}{\lfloor 5\, p^{7/2}}$$

$$- \frac{1}{\lfloor 7}\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{1}{2}\sqrt{\pi}}{p^{9/2}} + \cdots \infty, \text{ as } \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

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$$= \frac{\sqrt{\pi}}{2p^{3/2}}\left[1 - \frac{1}{4p} + \frac{1}{\lfloor 2}\left(\frac{1}{4p}\right)^2 - \frac{1}{\lfloor 3}\left(\frac{1}{4p}\right)^3 + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2p^{1/3}} e^{-\frac{1}{4p}}$$ #

First Shifting Theorem:-

If $L\{f(t)\} = F(p)$ when $p > a$, then

$L\{e^{at} f(t)\} = F(p-a)$, $p > \alpha + a$

i.e. If $F(p)$ is the Laplace transform of $f(t)$, then $F(p-a)$ is the Laplace transform of $e^{at} f(t)$.

Proof:- By definition, we have

$$F(p) = L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt$$

$$\Rightarrow F(p-a) = \int_0^\infty e^{-(p-a)t} f(t)\, dt$$

$$= \int_0^\infty e^{-pt} e^{at} f(t)\, dt.$$

$$= \int_0^\infty e^{-pt} \{ e^{at} f(t) \} dt$$

$$= L\{ e^{at} f(t) \}$$

Q. Find the Laplace transform of the following

(i) $L\{t^3 e^{-3t}\}$ (ii) $L\{e^{-3t} \sin 2t\}$ (iii) $L\{e^{t} \sin^2 t\}$

(iv) $L\{e^{2t}(3\sin 4t - 4\cos 4t)$

(v) $L\{e^{-t}(3\sinh 2t - 5\cosh 2t)$

Solⁿ:- (i) $L\{t^3 e^{-3t}\}$

$$\because L(t^3) = \frac{\lfloor 3}{p^4} = \frac{6}{p^4} = F(p)$$

From first shifting theorem

$$L\{e^{-3t} t^3\} = F(p-(-3)) = F(p+3) = \frac{6}{(p+3)^4}$$

(ii) $L\{e^{-3t} \sin 2t\}$

we have

$$\because L\{\sin at\} = \frac{a}{p^2+a^2}$$

$$\therefore L\{\sin 2t\} = \frac{2}{p^2+2^2} = F(p)$$

$\therefore$ By first shifting theorem

$$L\{e^{-3t} \sin 2t\} = F(p-(-3)) = F(p+3) = \frac{2}{(p+3)^2+4}.$$

(iii) $L\{e^t \sin^2 t\}$

We have $L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$

$$= \frac{1}{2} L\{1\} - \frac{1}{2} L\{\cos 2t\}$$

$$= \frac{1}{2}\frac{1}{p} - \frac{1}{2}\frac{p}{p^2+2^2} = \frac{1}{2}\left[\frac{\cancel{p^2}+4-\cancel{p^2}}{p(p^2+4)}\right]$$

$$\Rightarrow L\{\sin^2 t\} = \frac{1}{2}\frac{4}{p(p^2+4)} = \frac{2}{p(p^2+4)} = F(p)$$

Hence from first shifting theorem

$$L\{e^t \sin^2 t\} = F(p-1) = \frac{2}{(p-1)\{(p-1)^2+4\}}$$

$$= \frac{2}{(p-1)(p^2-2p+5)}$$

(iv) $L\{e^{2t}(3\sin 4t - 4\cos 4t)$

We have $L\{3\sin 4t - 4\cos 4t\}$

$$= 3L\{\sin 4t\} - 4L\{\cos 4t\}$$

$$= 3 \cdot \frac{4}{p^2+4^2} - 4\frac{p}{p^2+4^2}$$

$$= \frac{4(3-p)}{p^2+4^2} = F(p)$$