Reducing Algebraic Fractions – Rational Expressions

Note: This is typically the subject most elusive to students. They do not understand how fractions work, so Algebraic Fractions build a level of complexity on a non-existent foundation. Re-teaching fractions from the ground up has not been shown to be a worth-while investment in time. What does seem to work better is to allow issues that arise with Algebraic Fractions to make students receptive to addressing misconceptions.

Because this is elusive, and at this point students have minimal knowledge of Algebra, this is a topic that will be frequently revisited. When students learn how polynomial factoring works, this topic will be revisited with greater focus.

Be prepared to keep Algebraic Fractions on student radar weekly with homework, bell work, and quiz questions.

Big Idea

When reducing fractions, there are two basic fundamentals of mathematics at play. First, the commutative property of multiplication is being used extensively. Second, the fact that a non-zero number divided by itself is one is used. However, the process is simple to perform and difficult to write, so we rarely see it.

Key Knowledge

Students must be able to factor, including rewriting algebraic expressions with exponents.

Pro-Tip

When an expression has been reduced, it must have the same value as it originally. This can easily be verified by substituting integer values for unknown values.

In this section we will learn how Algebraic Fractions can be multiplied, reduced and added or subtracted. This particular entry will cover reducing and how reducing uses the greatest common factor of all terms.

It is often the case that students that once struggled with fractions gain insight and confidence with the rational numbers.

An algebraic fraction, or rational expression, is just a ratio of two algebraic expressions. The difference between an algebraic expression and a number is the variable, or unknown value. (Note that they're called expressions and not equations because they're not equal to anything.)

For example, $5x$, is the product of five and x. Since we do not know what x equals, we cannot carry out the multiplication. So, we just leave it written $5x$.

Another algebraic expression would be $15x^2$. This is the product of 15, x, and x.

An algebraic fraction, or rational expression of these two could be $\frac{5x}{15x^2}$.

This expression can be reduced and below we will see two ways to approach reducing algebraic fractions.

$$\frac{5x}{15x^2} = \frac{5x}{5x} \cdot \frac{1}{3x}$$

These are equal because when multiplying rational exponents you multiply the numerators together and then multiply the denominators together.

It is useful to separate $5x$ and $15x^2$ in this fashion because a number divided by itself equals one.

$$\frac{5x}{15x^2} = \frac{5x}{5x} \cdot \frac{1}{3x}$$

$$\frac{5x}{15x^2} = 1 \cdot \frac{1}{3x}$$

So this would be: $\frac{5x}{15x^2} = \frac{1}{3x}$.

To reduce you find what factors the numerator and denominator share and recognize that those shared factors are being divided by themselves, resulting in the number one.

The greatest common factor is what gets divided out of both the numerator and denominator. Another way to see this is below:

$$\frac{5x \div 5x}{15x^2 \div 5x} = \frac{\frac{5x}{5x}}{\frac{15x^2}{5x}} = \frac{1}{3x}$$

This method is less clear to see, but the math is the same.

Regardless of the method, the key piece of information required to reduce is the greatest common factor. The greatest common factor of two expressions is the largest expression that divides into the expressions in question.

For example: $3x^5y,\ 27xy$ and $9x^5y^2$ has a greatest common factor of 3*xy*, because 3*xy* is the largest thing that divides into all three terms.

Let's look at another example and use a table for factoring.

$15a^4b^3$, $20a^7b$, $30a^7b^2$

$15a^4b^3$	$20a^7b$	$30a^7b^2$
3•5•*a*•*a*•*a*•*a*•*b*•*b*•*b*	2•2•5•*a*•*a*•*a*•*a*•*a*•*a*•*a*•*b*	2•3•5•*a*•*a*•*a*•*a*•*a*•*a*•*a*•*b*•*b*

These are all of the factors of each of these expressions. To find the GCF we can make a list of the repeated factors, factors that are in common between all three expressions.

5• *a*•*a*•*a*•*a*•*b*

If we divide this GCF out of each term, we would be left with:

$15a^4b^3$

3•~~5•*a*•*a*•*a*•*a*•*b*~~•*b*•*b* = $3b^2$

$20a^7b$
2•2•~~5•*a*•*a*•*a*•*a*~~•*a*•*a*•*a*~~•*b*~~ = $4a^3$

$30a^7b^2$
2•3•~~5•*a*•*a*•*a*•*a*~~•*a*•*a*•*a*~~•*b*~~•*b* = $6a^3b$

Where this will come into play is with reducing something like:

$$\frac{15a^4b^3 - 20a^7b}{30a^7b^2}$$

By dividing out the GCF of all three terms we are left with:

$$\frac{15a^4b^3 - 20a^7b}{30a^7b^2} = \frac{3b^2 - 4a^3}{6a^3b}$$

Reducing:

To reduce an algebraic fraction all terms must have a common factor. Terms can be separated by the fraction bar or by addition or subtraction. The expression below has three terms, two in the numerator and one in the denominator. In order to reduce, all terms must share a common factor. What students will often do when reducing this expression is reduce the *a's*, just leaving the expressions of $-b$. Sometimes, they will realize that *a* divides into itself one time, so they will write $1 - b$.

Figure 1: $\frac{a-b}{a}$

Figure 2: $\frac{a-b}{a} = -b$

$$\text{Figure 3: } \frac{a-b}{a} = 1-b$$

If we assign some relatively prime numbers for *a* and *b*, and evaluate each of these figures we will see that they are not all equal. If the reducing was correct, each expression would be equal.

Let $a = 5$, $b = 3$

$$\text{Figure 1: } \frac{5-3}{5} = \frac{2}{5}$$

$$\text{Figure 2: } \frac{5-3}{5} = -3$$

$$\text{Figure 3: } \frac{5-3}{5} = 1-3 = -2$$

Only Figure 1 is correct. The others are incorrect because in order to reduce all terms must have a common factor. Here is why.

The order of operations governs the process in which we perform mathematical calculations. The fraction bar groups together the terms in the numerator, even though there are not any parenthesis. Operations grouped together must be carried out before any other operations. Division is reducing, which takes places after the group's operations are completed.

So why can we divide (reduce) before carrying out the group's operations? Consider the following example for an idea of why this is before reading the why this works.

$$\frac{15x+5}{10x}$$

The GCF of all the terms is five. This expression could be written as it is below.

$$\frac{15x+5}{10x}=\frac{5(3x+1)}{5\cdot 2x}$$

And we could write that as follows:

$$\frac{5(3x+1)}{5\cdot 2x}=\frac{5}{5}\cdot\frac{3x+1}{2x}$$

And five divided by five is one. The product of one and anything is, well, that anything. Multiplying by one does not change the value (that is why one is called Identity).

$$\frac{15x+5}{10x}=\frac{5(3x+1)}{5\cdot 2x}=\frac{3x+1}{2x}$$

The reason we can reduce before completing the operations in the group (numerator in this case), is because of the nature of multiplication and division being interchangeable. For example:

$$5\cdot 3\div 5=5\div 5\cdot 3$$

You may object here because in a previous section we showed how the order of division cannot be changed without changing the value. For example:

$$8\div 4=2$$

$$4\div 8=\frac{1}{2}$$

$$\frac{15x+5}{10x}=\frac{5}{5}\cdot\frac{(3x+1)}{2x}=\frac{3x+1}{2x}$$

Without going into too much detail, division is multiplication by the reciprocal, and there is multiplication by the same factor taking place in both numerator and denominator. So when reducing, you are simply dividing out that common factor before multiplying it, which is mathematically sound.

Regardless, it must be understood that to reduce an algebraic expression each term must contain a common factor. In the expression remaining from the example above, two of the terms contain a factor of x, but not the third. To reduce (divide), before adding the numerator together, would be in violation of the order of operations.

Practice Problems:

Reduce the following:

1. $\frac{32x^2y^4z}{14x^5y}$

2. $\frac{3a}{9a^2}$

3. $\frac{5x^my^3}{15x^my}$

4. $\frac{27a^2b+3a^2b}{99a^5b^3}$

5. $\frac{7xy-2}{4y^2z}$