Units

Ratio, Rate and Scale Factors

Note: This unit is on-going and will play a huge role in proportions in Geometry. However, the topic seems to find its way in multiple types of question on Cambridge IGCSE examinations, and is a great way to bolster student mathematical literacy.

This is just an introduction to the converting units.

Big Idea

Units are values that we attach to numbers which make them “real.” The number five is rather meaningless without a unit. Five bananas means ”*something real*.”

There are multiple ways to measure a distance, for example. When performing operations the units must be equal.

Key Knowledge

Students must know basic metric conversion rates. Here is a reference chart that can be used while learning. But this cannot be used on quizzes or tests, nor on the IGCSE examination.

Pro-Tip

When dealing with Paper 2 (no calculator), keep in mind the conversion calculation must be simple. Set up the conversions cleanly on paper and then reduce, reduce, reduce.

Well, hello there! Today is my birthday! Yup. I am 1 billion, three hundred and eighty seven million, five hundred and eighty four thousand second olds, well, as of 12 AM, you know, this morning. Now it’s 2:24 PM, so that’s 14 more hours, let’s see, 60 seconds per minutes, 60 minutes per hour … carry the four …

Units give us a reference point, a scale, so that numbers can have a meaning. If a calculator costs $24 we roughly know its value in relation a single dollar. That makes a lot more sense to someone that uses U.S. currency regularly than 2685 Yen, even though they are the same thing (as of right now anyway).

But, did you know that I am exactly 20 decks of cards tall? Yup, true story.

Why would I use a deck of cards as a unit? Well, sometimes it's really helpful, especially when determining a portion size of food. A serving of meat that is about a deck of cards in size is about 3 ounces of meat. There's a good scientific reason to use ounces for serving sizes, but they don't translate to ease of day to day use.

The point is that unit give numbers meaning by providing a reference. We know how long a foot is, how heavy a kilogram is and how hot 72 degrees Fahrenheit is. Without the units, numbers are abstractions. The number 517, for example, doesn't really mean anything real. I don't mean real as in the set of Real numbers, but it's just a number, not a number of something. We know things about 517 that will translate if we knew what we had 517 of, but otherwise that's just it, it's just a number.

Now units scale things for us, so we can comprehend them. With the birthday example, seconds are a totally invalid unit. Because my age, which if you can figure it out, let me know in the comments below, is such a large number of seconds it doesn't make sense. The scale is off. Years is what we need there.

However, if a truck, 3 meters long, is traveling at 120 kilometers per hour, and travels over a bridge that is 60 meters long, and we wanted to know how long it took for the truck to cross the bridge completely, we would probably have to use seconds. The number in terms of hours would be too small to make sense. The scale hours provides is inappropriate when you're talking about a few seconds. Similarly, the scale of millimeters would be too small if discussing the height of a mountain. (Note the unit for travel is a rate, or a ratio of two units. We'll get into that soon.)

Converting between one unit and the next doesn't need to be complicated. In most cases it is just an exchange of one value for an equivalent value. One must mind, however, the number base system being used. In the metric system, things are an easy power of 10, which is what our decimal system that calculators are programmed with, so life is easy.

But converting from inches to miles, or seconds to years gets tricky because those are not base 10 numbering systems. Still, we just need to break things down into sensible chunks and replaced appropriately.

Let's try an example. Let's say that Joe ran 7 miles at an average pace of 7:45 per mile. (Be careful here, that's not 7 hours and 45 minutes, but 7 minutes and forty five seconds.) What is the total time required to complete the 7 miles?

There are many completely efficient and correct ways to tackle this problem. You could separate the time 7:45 into 7 minutes and 45 seconds and then multiply each by the 7 miles. You could also turn the 45 seconds into a decimal, which is .75, not .45. Remember, 45 seconds is 45/60, while 0.45 is 45/100. Take care to make sure you understand what the numbers mean (their units).

There are other methods as well. For now, let's break the 7:45 apart, as follows.

To figure out how long 7 miles took, each mile taking 7:45, we need to multiply 7 × 7:45. This doesn't work on a calculator because of the 45 seconds. But we could perform the following:

$$7\ (7\text{ minutes} + 45\text{ seconds})$$

That would give us 49 minutes and 315 seconds. Beautiful.

Unfortunately, 315 seconds does not make sense. We need to convert that back to minutes and seconds. To do so, just divide by 60.

$$315 \div 60 = 5.25\text{ minutes}$$

(In just a moment we will discuss what happens with those units.)

But again, that 0.25 is 25/100, not 25/60, which would be seconds. No, it's a quarter of a minute, which is 15 seconds. Here's how we could calculate that if you didn't recognize the fact.

$$.25\text{ minute} \times 60 = 15\text{ seconds}$$

That tell us that 315 seconds is 5 minutes and 15 seconds. We know that we had 49 minutes of running time early, which brings us to 54 minutes and 15 seconds.

You might be objecting right now because I've been sloppy with my units. And it's true, I have been, but with intent. When I am converting seconds to minutes I am dividing by 60 because there are 60 seconds in one minute. When I am multiplying 0.25 times sixty, I am multiplying the minutes (0.25) by how many seconds in each minute.

Sometimes we write out too much math and get ourselves confused, or just plain overwhelmed by the amount of information written. We can, in my opinion, be too careful on paper and cause exactly the thing we hope to avoid, mistakes.

Here is an explanation of what happens to the units. This is very important, and depending on the level of complexity of the problem being handled, I would highly recommend writing out the units. But for time conversion, seconds to hours, I don't think it's necessary.

Let's do the following first, because it's simplest.

$$.25 \text{ minute} \times 60 = 15 \text{ seconds}$$

This is really:

$$\frac{0.25\,\text{minutes}}{1} \times \frac{60\text{ seconds}}{\text{minutes}} = \frac{0.25 \times 60 \times \text{minutes} \times \text{seconds}}{\text{minutes}}$$

Something divided by itself is one, unless its zero. Then, minutes divided by minutes reduces to one and we are left with the product of 0.25 and 60, with a unit of seconds.

I think the conversion, or scale factor, of 60 seconds to a minute is simple enough to account for by memory.

Now the first conversion has some math that might be confusing, but it is imperative that complex fractions are understood.

$$315 \div 60 = 5.25 \text{ minutes}$$

This is really:

$$315 \text{ seconds} \div 60 \text{ seconds/minute}$$

That's division of division! Let's rewrite it in a cleaner fashion.

$$\frac{315\text{ seconds}}{1} \div \frac{60\text{ seconds}}{\text{minute}}$$

The way we divide is to multiply by the reciprocal.

$$\frac{315\text{ seconds}}{1}\cdot\frac{\text{minutes}}{60\text{ seconds}}=\frac{315}{60}\times\frac{\text{seconds}}{\text{seconds}}\times\frac{\text{minutes}}{1}$$

That's how we end up with 5.25 minutes.

Time is particularly trick to deal with because we often need more than one unit in our answer. This is similar to distances measured in feet and inches, like how tall you are. But sometimes, if we know the unit required for our answer, we can translate all of the units into that required unit before computing anything else and life is easy!

Example:

Sue drinks 8 water bottles a day. Each bottle holds 750 milliliters. (Sue has kidney failure, that's why she drinks so much.) How many liters of water does she consume daily?

The calculation to be performed is just:

$$8 \times 750\text{ ml} = ???$$

But if we convert milliliters to liters first, we can convert the 750 ml into liters, we are making a conversion with 750 instead of eight times that much, the numbers being smaller are easier.

There are 1,000 milliliters to 1 liter, so the following proportion can be established.

$$\frac{1\text{ liter}}{1000\text{ ml}}=\frac{x\text{ liter}}{750\text{ ml}}$$

While the numbers here may lend themselves to a quick and easy calculation, this also makes them useful for something more powerful, the formulation of a method.

If you remember, a ratio compares two things, like liters to milliliters, as written on the left and the right. Proportions are equal ratios. Notice that the unknown quantity was written in the numerator, which makes it easier to solve for that quantity. If we were converting to milliliters it would be best to write these ratios with liters in the denominator.

750 ml is .75 liters. Substituting this into our calculation:

$$8 \times 750 \text{ ml} = ???$$

$$8 \times 0.75 \text{ l} = 6 \text{ liters}$$

Again, the point of writing out the work is to establish a procedure that can be used when the numbers are less friendly.

Now, what about something a little trickier, like that truck traveling at 120 kilometers per hour?

I will not work out that problem in this section, but will leave it to you. However, I'll give you a few clues and point you in the right direction. First, realize that at 120 kilometers an hour it will take but seconds to pass over the bridge. Our unit of time will need to be in seconds, most likely. The second thing is the distance of the unit is in kilometers, but the bridge is in units. Those need to be the same unit, so the scales match. Either the 120 kilometers needs to be converted to meters, or the 60 meters needs to be converted into kilometers. The choice is yours, but be wise about it. Last thing about the truck problem. Do you see that you're given how long the truck is? (The problem is below.) Do you also see how it asks for the time it takes the truck to completely cross the bridge?

One last word, if I may. All of these conversions are linear. In the next section we will see how units and scale factor must be addressed when dealing with non-linear units like area and volume.

Practice Problems

1. A truck, 3 meters long, is traveling at 120 kilometers per hour, and travels over a bridge that is 60 meters long. How long will it take for the truck to cross the bridge completely?

2. There are twenty eight days in February. How many seconds is that?

3. Bob is 6’ 1” tall, which is 21 decks of cards tall. An inch is 2.54 centimeters. How many centimeters tall is a deck of cards?