Online Appendix: proofs of illustrative examples

February 26, 2021

1 Introduction

Pachelly and Los Chatas

Notation

Subscript \(i\) denotes each of the two rival groups: Pachelly (\(P\)) and Los Chatas (\(C\)). \(-i\) is just the notation assigned to the rival of \(i\). \(U_i(\cdot)\) is the utility group \(i\) gets from taking an action (either choosing war or peace).

In turn, \(R\) is a generic way to denote the total value/size of the pie, \(s_i\) is the share of \(R\) controlled by \(i\), \(c_i\) is the cost of war for group \(i\), and \(x\) is the amount of the transfer that \(-i\) has to give to \(i\).

Setup

Imagine Bello, on the northern edge of Medellin, is worth $100 in extortion, drug corners, and other criminal rents. And suppose that the two main criminal groups, Pachelly and Los Chatas, each start with an equal share of the neighborhood ($50 each).

Let’s also say that, militarily-speaking, Pachelly and Los Chatas are evenly matched. This means, if it comes to war, each gang has an equal chance of winning — 50 percent. Assume the gang that wins the fight gets the whole of Bello forever; the loser gets nothing. The two rivals know, as we do, that war has dire consequences—killing soldiers, destroying business, and attracting police attention. In the struggle for Bello, let’s suppose each side knows that fighting will cost them $10, win or lose.

Claims

1. Tom would happily choose peace so long as the deal gives Los Chatas control of at least 40 percent of Bello. Pachelly faces identical incentives, and Tom and his Los Chatas know it. Pachelly will also be hap-pier with 40% of the pie over war. Neither side will fear an attack, and so neither side will provoke a war.

2. The bargaining range also means that deals don’t have to be equal or stable to be durable. Imagine that Tom’s lieutenants outmaneuvered Pachelly and occupied a series of drug corners without a shot fired. Should Pachelly launch an attack to get back their plazas? So long as Los Chatas controlled less than $60 of Bello in total, Pachelly would grumble and posture but ultimately resign themselves to the new reality. War is damaging enough that a certain and peaceful $41 would still be a better option than fighting.

3. What happens if the balance of military power shifts in Medellín? Imagine that Pachelly makes an honest assessment of its new strength and figures its chance of winning has fallen from 50 to 25 percent. Now Tom and Los Chatas could now win a war 75 percent of the time. Nothing else has changed so far, and so Tom and Los Chatas still occupy half of Bello.
Regardless, Tom will not launch a war and Pachelly will prefer any peaceful deal that leaves them at least with 15% of Bello. The bargaining range where both sides has shifted, from one where Pachelly is content with $40 to $60, to one where they’ll grudgingly accept $15 to $35.

Proofs

1. If group $i$ chooses war, she flips a coin for the pie ($\$100$) and suffers $\$10$ from fighting. In expectation, that’s worth $\$40$. This is true for both, Pachelly, $P$, and Los Chatas, $C$.

$$E[U_i(war)] = (0.5 \times 100 + 0.5 \times 0) - 10 = 40 \forall i = C, P$$

Both sides are always better off finding a compromise, i.e., the status quo. As long as war is costly, it shouldn’t make sense to wage it. Both sides know it. To see this, consider the options facing Tom, the leader of Los Chatas and one of the city’s criminal kingpins. If Tom chooses peace and the status quo, Los Chatas maintain $\$50$ of territory. But if he chooses war, he flips a coin for the pie and suffers $\$10$ from fighting. In expectation, that’s worth no more than $\$40$ to Tom. Pachelly faces the same choice, and Tom and Los Chatas know it. Each gang grasps that a sure bet of $\$50$ is better than a risky shot at $\$40$.

$$E[U_i(peace)] > E[U_i(war)]$$

$50 > (0.5 \times 100 + 0.5 \times 0) - 10 = 40 \forall i = C, P$

2. A cool-headed, calculating Pachelly will resign themselves. War is damaging enough that a certain and peaceful $\$41$ is still a better choice than the $\$40$ expected value of war.

$$E[U_P(resign)] > E[U_P(war)]$$

$41 > (0.5 \times 100 + 0.5 \times 0) - 10 = 40$

3. War nets the weakened Pachelly no more than $\$15$. So, Pachelly will prefer any peaceful deal that leaves them at least $\$15$ in territory. The bargaining range where both sides has shifted, from one where Pachelly is content with $\$40–60$, to one where they’ll accept $\$15–35$ without war. Pachelly still wants to make a deal—to offer Los Chatas just enough of Bello to make peace more attractive than war. Why not cede to Los Chatas some territory? Or pledge tribute every year? If Pachelly relinquished just $\$16$ in territory, Los Chatas would prefer peace to war.

This is because war offer Los Chatas just $\$65$ in expectation. Pachelly would love to be able to make a take-it-or-leave-it offer of $\$66$. That would leave Pachelly with a healthy $\$34$ in territory—far more than the $\$15$ they could expect to win through battle. Likewise, if Los Chatas could make Pachelly a take-it-or-leave it offer, they would offer Pachelly the minimally acceptable offer. This defines the new bargaining range, where neither side wants to fight. Thus, a peaceful deal is one that leaves Pachelly anything between $\$15$ and $\$35$.

More formally, we have that for Los Chatas

$$E[U_C(bargain)] \geq E[U_C(war)]$$

$$0.5 \times 100 + x \geq 0.75 \times 100 + 0.25 \times 0 - 10$$

$$x \geq 15$$
In turn, for Pachelly

\[ E[U_P(bargain)] \geq E[U_P(war)] \]
\[ 0.5 \times 100 - x \geq 0.25 \times 100 + 0.75 \times 0 - 10 \]
\[ 35 \geq x \]

Hence, \( x \in [15, 35] \).

## 2 Chapter 1: Unchecked private interests

### Colonists and the Crown

**Notation**

Subscript \( i \) denotes each of the two rival groups: the Crown (\( C \)) and George Washington (\( W \)). \( U_i(\cdot) \) is the utility group \( i \) gets from taking an action (either choosing \( \text{war} \) or \( \text{peace} \)).

In turn, \( R \) is a generic way to denote the total value/size of the pie, \( s_i \) is the share of \( R \) controlled by \( i \), \( c \) is the cost of war, \( a \) is the fraction of the cost of war that Washington bears, \( y \) is an element of the set shares of the pie that would induce Washington to prefer peace. Finally, \( b \) is the extra benefit that Washington gets by winning the war with respect to the rest of the colonists who also won.

**Setup**

Imagine that all the 13 colonies’ land, taxes, and other spoils are a pie that the British Crown and the American colonists must share between them. Now it was time for the colonists to pay their share of their continent’s defense and administration. And so, the Crown began to levy taxes.

The Colonists were outraged. Why should they pay more, when they haven’t a formal say in this faraway government? But then, why didn’t the Crown grant the Colonists representation in exchange for more taxes? Revolution is puzzling because fighting would be long and brutal. These costs—\$10 to each side opening up a range for bargaining \$20 wide. Assuming leaders weigh all the costs and benefits, both sides have more to profit from a peaceful split of the \$100 colonial pie. The key phrase here is “assuming leaders weigh all the costs and benefits”. Economists call this the “unitary actor” assumption. It means the ruler is trying to maximize the group’s collective interests, not their own.

Assume the colonists and the crown have equal power, that is, they are equally likely to win a war.

### Claims

1. Assume Washington and the Crown leaders weigh all the costs and benefits of war. Then both parts will accept a peaceful split offering more than 40\% of the pie.

2. Dispense now the unitary actor assumption. Let Washington alone chooses whether the colonies go to war against the crown. Additionally, suppose now that Washington only bears a fraction, \( a \in [0, 1] \), of the costs of war. Then, there could be some circumstances under which war is preferable.

### Proofs

1. The assumption requires George Washington to weigh the costs and benefits to his citizens as if these were pains and pleasures of his own. A benevolent Washington, for example, would curb his bottomless appetite for land. He would relinquish precious Western territories to Canada. He would do this all for the greater good. Likewise, on the Crown side, King George and the aristocrats that ran England’s Parliament would look beyond their own interest and consider the lowly soldier, or the disruption to
Britain’s merchants (none of whom had the vote). That’s because, for closely matched adversaries like the 13 Colonies and Britain, going to war is a risky, costly wager, akin to each side flipping a deadly coin. Heads they win it all, tails they lose everything. As you might remember, this violent gamble is worth no more than $40 in expectation—a 50% shot at the pie, minus the costs from fighting. For a unitary actor—the ultimate benevolent dictator—any peaceful split that offers more than 40% of the pie is a better deal than war.

\[
E[U_i(\text{peace})] > E[U_i(\text{war})]
\]

\[
E[U_i(\text{peace})] > (0.5 \times 100 + 0.5 \times 0) - 10 = 40 \quad \forall i = C, W
\]

Where \( C \) corresponds to the Crown and \( W \) to George Washington. Hence, both Washington and the Crown would accept the peaceful split if it ends up offering them more than 40% of the pie.

2. We just need to find an example where war is preferable for Washington, so let us assume that if he chooses peace, he will be able to claim 30 percent of the Colonist’s share of the pie. If he chooses war, however, victory brings him personal benefits. Winning the war will let him claim a greater share of the pie—50 instead of 30 percent.

\[
E[U_W(\text{peace})] = 0.3 \times y
\]

\[
E[U_W(\text{war})] = 0.5 \times (0.5 \times 100 + 0.5 \times 0) - c_W = 25 - c_W
\]

Where \( y \in [40, 60] \) is any of the possible settlements that would prevent war. We can see then that Washington will declare war if \( 25 - 0.3y > c_W \); equivalently, if \( c_W \in [7, 13] \). To fix ideas, if we just assume the original endowment of \( y = 50 \), then \( E[U_W(\text{peace})] = 15 \). So, unless the costs of war for Washington are at least as high as those for all the colonists (10), he will declare war.

In this case, let us say that \( c_W = a \times 10 \), and assume that \( y \) is equal to the original endowment of 50.

\[
E[U_W(\text{peace})] = 0.3 \times 50 = 15
\]

\[
E[U_W(\text{war})] = 0.5 \times (0.5 \times 100 + 0.5 \times 0) - a \times 10 = 25 - a \times 10
\]

Hence, Washington will declare war if \( 1 > a \). Since \( a \in [0, 1] \), then Washington will only prefers peace if \( a = 1 \), in any other case he prefers to declare war.

3 Chapter 2: When Violence is Valued

Campesinos and Elites

Notation

Subscript \( i \) denotes each of the two rival groups: Campesinos (\( C \)) and Elites (\( E \)). \( -i \) is just the notation assigned to the rival of \( i \). \( U_i(\cdot) \) is the utility group \( i \) gets from taking an action (either choosing war or bargain).

In turn, \( c \) is the cost of war, and \( x \) is the amount of the transfer that \( -i \) has to give to \( i \). Finally, \( v \) represents the rightful vengeance benefit for the Campesinos.

Setup

Suppose the pie is control of El Salvador’s vast coffee haciendas. Instead of gangs or British colonials, however, our two sides are now peasants and elites. The dispossessed campesinos have organized themselves for the first time. They’re a threat to the oligarchic order, with even odds of victory. The elites have a choice. They can concede to peasant power, break up some of the biggest estates into cooperatives, but
still hold on to half the land. Or they can fight and try to keep it all. Victory would cement their system of haciendas and oppression, minus the costs of war. With material costs of war of $10 to each side, the bargaining range is $20 wide. This should be ample room for land reforms and representation to keep the campesinos from revolt.

Once upon a time, it seemed like serfdom was the natural order of things and campesinos were treated no better than animals. Hence, now introduce a violent value: righteous outrage. While anger sweeps peasants across the country. The emotional rewards (call it $10 in righteous pleasure) completely offsets campesinos’ costs of war.

Claim

Peace will still be preferred over war.

Proof

The bargaining range has shrunk by half. This violent value acts much like war profits did in the last chapter: they give one side an explicit incentive to fight. Still, note that even with half the bargaining range gone, we continue to predict peace. The elite’s cruelty have simply eroded their bargaining power. Now the peasants will accept no less than half the country’s lands, and prefer fighting otherwise. But elites still feel the costs of war, leaving a 10-dollar wedge. Ceding 40 to 50 percent of the land is still more attractive than fighting. With the bargaining range shrunk, however, now there is less room for error—errors that could come from the other drivers of conflict, such as an unchecked autocrat, or uncertainty. So at the very least, emotional reactions (like righteous vengeance) or a desire for glory make peace more fragile. The new bargaining range is given by the following two inequalities. First, for the campesinos we have that

$$E[U_C(\text{bargain})] \geq E[U_C(\text{war})]$$

$$0.5 \times 100 + x \geq 0.5 \times 100 + 0.5 \times 0 - 10 + 10$$

$$x \geq 0$$

In turn, for the elites

$$E[U_E(\text{bargain})] \geq E[U_E(\text{war})]$$

$$0.5 \times 100 - x \geq 0.5 \times 100 + 0.5 \times 0 - 10$$

$$10 \geq x$$

Therefore, $$x \in [0, 10]$$. In other words, the bargaining range is between 0 and plus 10 of the original wealth of each of the groups. Notice then that only the peasants can benefit from the new bargaining range. That is, the campesinos have all the bargaining power.

4 Chapter 3: Uncertainty and information problems

Lords and Stones

Notation

Subscript $$i$$ denotes each of the two rival groups: Vice Lords ($$Lo$$) and the Stones ($$St$$). $$-i$$ is just the notation assigned to the rival of $$i$$. $$U_i(\cdot)$$ is the utility group $$i$$ gets from taking an action (either choosing war or peace).

In turn, $$x$$ is the amount of the transfer that $$-i$$ has to give to $$i$$. 
Setup: Uncertainty as noise (Different Beliefs)

Let’s consider what happens when two gangs have different predictions about who will win a conflict. Take the case of the Vice Lords who believed they were well matched against the Stones. Let’s imagine the Lords were looking at the familiar-looking $100 pie from the previous sections, where the bargaining range is between $40 and $60. But suppose the Stones saw little bitty Nap Dog (the Lords leader), the untested 17-year old chief, and figured that times have changed. Suppose they thought they’ll win three-quarters of the time if it comes to a fight, leaving the bargaining range between $15 and $35. Of course, as usual, the cost of war is $10.

Claim

Under these circumstances there is no overlap in the bargaining ranges and we predict war.

Proof

The Vice Lords are willing to accept as little as $40 of Horner to avoid war. But the Stones won’t give the Lords a penny more than $35. There’s no overlap in their bargaining ranges. Previously, the pie-splitting was peaceful for the simple reason that both sides held the same prior beliefs. When we relax this “common priors” assumption, fighting becomes a sort of learning-by-doing. This is an extreme example, with no overlap at all. If we picked a less extreme case, where the two rivals’ priors aren’t quite so far apart, then the bargaining ranges will intersect. But that range is narrower than if they’d started with the same reading of the situation. Hence, if the bargaining ranges intersect, then we should expect some space for a peaceful pie-splitting. Otherwise, war would be inevitable.

In particular, Lords, \(L\), decide to bargain if

\[
E[U_L(\text{peace})] \geq E[U_L(\text{war})]
\]

\[
50 + x \geq (0.5 \times 100 + 0.5 \times 0) - 10 = 40
\]

Where \(x \in [-10,10]\) is the transfer between the Lords and the Stones, which at the same time defines the expected bargaining range for the Lords, since they assign a belief, \(\mu_L\), equals to 1 to the fact that they are equally strong as the Stones (see question 2.4 to see how the bargaining range is derived). In turn, the Stones assign a belief, \(\mu_S\), equal to 1 to the event that the Lords are weaker than them, thus assigning 0.75 to the probability that they can defeat the latter. Therefore, the Stones will prefer peace over war only if

\[
E[U_S(\text{peace})] \geq E[U_S(\text{war})]
\]

\[
50 - x \geq (0.75 \times 100 + 0.25 \times 0) - 10 = 65
\]

Hence, the minimum amount the Stones would need to prefer peace over war is 65. Now, suppose that in order to sustain peace, the Lords transfer \(x = 15\) to the Stones. We immediately see that \(x \leq 10\) in order for the Lords to prefer peace over war. Therefore, \(x = 15\) is a contradiction and peace is not possible to sustain.

Uncertainty and Bluffing

Notation

Subscript \(i\) denotes each of the two rival groups: Vice Lords (Lo) and the Stones (St). \(-i\) is just the notation assigned to the rival of \(i\). The Lords could be of one out of two possible types, which are given by \(s\) (strong) and \(w\) (weak). So, the Lords will be identified by a pair \((\cdot, Lo)\), where the first entry could be either \(w\) or \(s\). \(U_i(\cdot)\) is the utility that group \(i\) gets from taking an action.
In turn, $\mu$ is the Stones belief that the Lords are strong. Actions for the Lords are $T$ to make a threat or $NT$ to not make a threat. The actions for the Stones are $L$, offering a low amount of money, or $H$, offering a high amount of money.

$p$ is the probability that the Stones plays $L$, and $q$ is the probability that the weak Lords ($w,Lo$) plays $T$.

Finally, a Bayes-Nash Equilibrium (BNE) is given a by a belief and an action (possibly a mixed strategy) for each player’s type in the following form: $((\cdot,\cdot),\cdot)$. Where the first entry of the inner parenthesis is the strategy of player ($w,Lo$), the second entry of the inner parenthesis is the strategy of player ($s,Lo$), and the second entry of the outer parenthesis is the strategy of the Stones.

Setup: Private Information

Now, the uncertainty takes a different, simpler form: Are the Vice Lords weak or strong? Suppose the old men at the top of the Vice Lords know a bitter truth—they are sapped and outmatched at Horner. They figure they have a three-quarters chance of losing a war against the Stones. After all, their local leader is 17. And the Stones just raided their building with Uzis!

But there is hope. That’s because the Stones aren’t certain of the truth. The Stones assign some chance that the Lords have grown weak, and some chance they’re just as strong as they originally were (where they were as strong as the Stones). Only the Lords know the reality—they have what’s called “private information”. The Stones know that both scenarios are possible. They also know that the Lords have an incentive to bluff, and they regard every signal suspiciously.

As usual, assume the cost of war is $10 and the total pie is worth $100. Also, assume that if a bluff is called, it results in a war. Otherwise, the result is peace.

Claim

Assume the Stones belief that the Lords are strong is given by $\mu$, which is a distribution over the possible types assigned to the Lords (states of the world). Further assume that (i) the actions for the Lords are to make a threat ($T$) and to not make a threat ($NT$), the actions for the Stones are to offer a low amount of money ($L$) and to offer a high amount of money ($H$), and (ii) if the Lords are strong, they will always prefer to make a threat ($T$). Then, there exist then an equilibrium where peace is sustained and another one where war can occur.

Proof

NB: Notice that this is a sequential game. But as any sequential game, it has a normal form representation (i.e. simultaneous-kind of game). We will analyze this game as the latter, as it is usual for this kind of settings.

In this game, we have two states of the world. In the first one, the Lords are strong ($s$), and in the second one, the Lords are weak ($w$). Let us now define the different payoffs under the different strategies.

If the Lords are strong, we know they will always choose $T$. If the the Stones choose $L$ (trying to call a potential bluff), then both sides go to war and get 40 in expected value. If the Stones choose $H$, they will offer just enough to make the potential strong type to accept the deal, that is Lords get 41 and Stones get 59.

If the Lords are weak and they choose $NT$, then the Stones will immediately know that the Lords are weak since if they were strong, they would never choose $NF$. Hence, $\mu(w|NT) = 1$ and $\mu(s|NT) = 0$. Consequently, it will always be optimal for the Stones to play $L$, so that Lords get 16 and Stones get 84; any other choice from the Stones would give them strictly less payoff. (For the sake of completeness, we could say that for $(NT,H)$ the payoff would be 41 for the Lords and 59 for the Stones).
If the Lords are weak and they choose T, they will be trying to be pooled as a strong type (they will be bluffing). So, if the Stones choose L (trying to call the bluff), war will occur and since the Lords are weak the expected payoffs would be 15 for the Lords and 65 for the Stones. If the Stones decide to be cautious, they will choose H (to offer a high amount of money for the settlement). Thus, the Lords would get 41 and the Stones 59. In this case, we have that $\mu(s|T) = \mu$ and $\mu(w|T) = 1 - \mu$.

Figure 1 summarizes the payoffs.

**Table**

<table>
<thead>
<tr>
<th>(strong) Lords</th>
<th>Stones</th>
<th>Offer Low (L)</th>
<th>Offer High (H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Make a Threat (T)</td>
<td>40,40</td>
<td>41,59</td>
<td></td>
</tr>
<tr>
<td>Not make a Threat (NT)</td>
<td>-,-</td>
<td>-,-</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(weak) Lords</th>
<th>Stones</th>
<th>Offer Low (L)</th>
<th>Offer High (H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Make a Threat (T)</td>
<td>15,65</td>
<td>41,59</td>
<td></td>
</tr>
<tr>
<td>Not make a Threat (NT)</td>
<td>16,84</td>
<td>41,59</td>
<td></td>
</tr>
</tbody>
</table>

From now on, Lo will stand for the Lords and St will stand for the Stones.

**Conditions for peace in equilibrium**

We define a BNE as $((\text{Strategy for Weak Lo}, \text{Strategy for Strong Lo}), \text{Strategy for St})$.

First, notice that since strong Lo always choose T, then if the Lo choose NT they are immediately revealing their type. Therefore, if St observe NT, they choose L, and notice that if Lo deviates to T, they will only get 15 instead of 16. Since they do not have any profitable deviations, $((\text{NT},T),L)$ is a Bayes Nash Equilibrium (BNE) sustaining peace.

Second, if Lo chooses T, St will choose H (and we would get peace as an outcome) if

$$E[U_{St}(L,T)] \leq E[U_{St}(H,T)]$$

$$\mu 40 + (1 - \mu) 65 \leq \mu 59 + (1 - \mu) 59$$

$$\frac{6}{25} \leq \mu$$

Now, fix St choosing H, we know the strong type will always choose T, and if the weak type deviates to NT, then she will get 41, which is exactly what they get under $(T,H)$. Therefore, if $\mu \in \left[\frac{6}{25}, 1\right]$, St chooses H and Lo chooses T since they do not have any profitable deviation. Consequently, peace would prevail.

A BNE assigns an action to each type of player. Hence, as we saw above, $((T,T),H)$, with $\mu \in \left[\frac{6}{25}, 1\right]$ is a BNE of this game. Notice that since both types of the Lords play T, then we have found a pooling equilibrium.

Also, as we have seen in the previous question, $((\text{NT},T),L)$ is a BNE separating equilibrium since each type of Lo plays a different strategy.

Finally, we need to find the mixed strategy BNE. Assume $p$ is the probability that St plays L and $q$ is the probability that the weak Lo, $(w,Lo)$ plays T. The Lo will mix if

$$E[U_{(w,Lo)}(T)] = E[U_{(w,Lo)}(NT)]$$
\[ p15 + (1 - p)41 = 16 \]
\[ \frac{25}{26} = p \]

Notice that the expected value of choosing NT is always 16 since St always choose L when they see NT.

Now, the St will mix if
\[ E[U_{St}(L)] = E[U_{St}(H)] \]
\[ \mu40 + (1 - \mu)[q65 + (1 - q)84] = \mu59 + (1 - \mu)59 \]
\[ q = \frac{44\mu - 25}{19(1 - \mu)} \]

So, under mixed strategies the BNE is \(((\frac{44\mu - 25}{19(1 - \mu)})T + (1 - \frac{44\mu - 25}{19(1 - \mu)})NT, T), \frac{25}{26}L + \frac{1}{26}H\).

Notice that this is the unique case where war may arise in our setting.

5 Chapter 4: Commitment problems

Sparta and Athens

Notation

Subscript \(i\) denotes each of the two rival groups: Sparta (\(S\)) and Athens (\(A\)). \(-i\) is just the notation assigned to the rival of \(i\). \(U_i(\cdot)\) is the utility group \(i\) gets from taking an action (either choosing war or bargain).

In turn, \(x_t\) is the amount of the transfer that Athens has to give to Sparta in period \(t\), \(BR\) is the set of best response actions for a player.

Finally, a Subgame Perfect Nash Equilibrium (SPNE) is of the form: \{\(\cdot, \cdot, \cdot\}\}. Where the first entry of \{\} is given by the strategy of Athens at period 0 and at period 2, respectively, and the second entry of \{\} is given by the strategy of Sparta at period 1.

Setup

Turning the classical Greek world into a now familiar $100 pie, let’s suppose that at the outset of the 5th century BC (around the time the Persians were expelled from mainland Greece) Sparta and its Peloponnesian League could win a war against Athens and its allies 75% of the time. And assume a cost of war of $10.

Athens then begins its ascent and at the same time, Sparta also suffered setbacks. As a result, by the middle of the 5th century, suppose that Sparta could foresee a day when the balance of power would be more even where both sides have 50% chance of winning a war and each having 50% of the pie. A crucial detail is that this shift from a 75-25 to a 50-50 match in military power hasn’t come about, yet. The rebalancing could be averted if Sparta goes to war and wins.

Let’s imagine there are two periods: today and the future. Sparta and Athens are not just bargaining over today’s $100 pie, the contest is also for $100 in the future. Also, assume there is a discounting factor equal to 1.

Claims

1. Sparta needs a promise of at least $140 not to invade. That demand looks feasible. Knowing it can get at least $40 in the future (maybe as much as $60) all Sparta needs today is the balance—between $80 and $100. That’s a high price for Athens today—it means wagonloads of tribute, a whole colony, or other exorbitant concessions. But peace is conceivable. There’s no fundamental commitment problem here, despite a huge expected
shift in relative strength. Sparta will believe that Athens will do this transfer and peace is going to be the equilibrium.

2. Keeping the chances of winning for Sparta in period 1 as before, but now considering a power shift in period 2 where now Athens has a 75% chances of winning. Now, Sparta will not believe that Athens will do the transfer that guarantees peace, so war is going to be the equilibrium.

Proof

1. If Sparta attacks Athens today, it knows has a 75% chance of getting both today and tomorrow’s pies, minus today’s $10 cost of war. That option is worth $140 to Sparta. Meanwhile, Athens expects to win just 25% of the time, so its expected value of war is just $40.

\[
E[U_S(\text{war})] = 0.75 \times (100 + 100) - 10 = 140\]
\[
E[U_A(\text{war})] = 0.25 \times (100 + 100) - 10 = 40
\]

Where S stands for Sparta and A for Athens. Notice that Athens’ 50% future chance of victory isn’t relevant in today’s decision, because the war is being fought now, when the balance of power is 75-25.

We can derive the bargaining range as follows:

For Sparta in period 1 we have that

\[
E[U_S(\text{bargain})] \geq E[U_S(\text{war})]
\]
\[
0.75 \times 100 + x_1 \geq 0.75 \times 100 + 0.25 \times 0 - 10
\]
\[
x_1 \geq -10
\]

In turn, for Athens

\[
E[U_A(\text{bargain})] \geq E[U_A(\text{war})]
\]
\[
0.25 \times 100 - x_1 \geq 0.25 \times 100 + 0.75 \times 0 - 10
\]
\[
10 \geq x_1
\]

Therefore, \(x_1 \in [-10, 10]\). In other words, the bargaining range is between minus 10 and plus 10 of the wealth of each of the groups in period 1. Hence, between 65 and 85.

For Sparta in period 2 we have that

\[
E[U_S(\text{bargain})] \geq E[U_S(\text{war})]
\]
\[
0.5 \times 100 + x_2 \geq 0.5 \times 100 + 0.5 \times 0 - 10
\]
\[
x_2 \geq -10
\]

In turn, for Athens

\[
E[U_A(\text{bargain})] \geq E[U_A(\text{war})]
\]
\[
0.5 \times 100 - x_2 \geq 0.5 \times 100 + 0.5 \times 0 - 10
\]
\[
10 \geq x_2
\]

Therefore, \(x_2 \in [-10, 10]\). In other words, the bargaining range is between minus 10 and plus 10 of the wealth of each of the groups in period 2. Hence, between 40 and 60.
This means there is still a bargaining range $20 wide—the same as the total cost of war. Notice
that the future matters, because it limits what kinds of peaceful deals are credible now. As we men-
tioned above, any deal that leaves Sparta with $140 to $160 in expectation, both sides will prefer to
peace.

Observe then that once the power shift happens, we are in the world of the second pie, where it
never makes sense for Athens to give Sparta a penny more than $60. War is preferable to any further
sacrifice. Indeed, Athens might hope to get away with as little as $40, the lower end of the new bar-
gaining range. Therefore, Sparta looks ahead and expects its share to be somewhere between the two
amounts.

Sparta knows all this today, when it needs a promise of at least $140 not to invade. That demand looks
feasible. Knowing it can get at least $40 in the future (maybe as much as $60) all Sparta needs today
is the balance—between $80 and $100. That’s a high price for Athens today—it means wagonloads of
tribute, a whole colony, or other exorbitant concessions. But peace is conceivable.

In particular, observe that Athens can at most (any additional penny would make Athens incentives
for war to be greater than those for peace) leave Sparta with as much as $85 in period 1, then in period
2 Athens only needs to leave Sparta with at least $55. In turn, since in period 2 is only credible for
Athens to leave Sparta with as much as $60 (more would induce Athens to prefer war), then Athens
should at least leaves Sparta with $80 in period 1. Therefore, we conclude that two-periods bargaining
range, BR, is given by (following the notation from the previous question):

$$
BR = \{(x_1, x_2) : x_1 \in [5, 10], x_2 \in [5, 10], x_1 + x_2 \geq 15\}
$$

There’s no fundamental commitment problem here, despite a huge expected shift in relative strength.

**Period 0:** Athens, A, makes a take-it-or-leave-it offer $x_1$. The two actions at this node are: $x_1 \equiv x_1 \in [5, 10]$ or $x_1 \in [5, 10]$. If A chooses the latter, the game ends in war where the expected payoffs are (40, 140) for Athens and Sparta, respectively. If A chooses the former, the game continues.

**Period 1:** Sparta, S, accepts or rejects $x_1$. If Spartans accept it the game continues. If they reject it the game ends in war where the expected payoffs are (40, 140).

**Period 2:** Athens offers $x_2$. Where the actions are $(x_1, x_2) \in BR$ or $(x_1, x_2) \notin BR$. Where BR was defined in the previous answer. If $(x_1, x_2) \in BR$ is chosen, the outcome is peace with (60, 140) as the payoff. If $(x_1, x_2) \notin BR$ is chosen, we have war with (25 - $x_1 + 40$, 75 + $x_1 + 40$) as the outcome. Notice that in the latter payoff there was peace in period 1 and war occurs only in period 2. Thus, we add the expected value of war ($40) to the peace payoffs of period 1. Notice that $25 - x_1 + 40 \leq 60$.

By backwards induction, and since Sparta has no profitable deviation, the SPNE is given by

$$\{(x_1 \in [5, 10], (x_1, x_2) \in BR); Accept\}.$$

2. In the future, the balance of power will flip until Athens has the advantage. Sparta is still dominant
today (the first pie), and so it still needs $140 to avoid war. But once the new balance of power comes
about, Athens can reliably promise just $15 to $35 to Sparta. To be better off at peace than at war, Sparta needs the balance today—at least $105 worth, if not more. That’s bigger than the entirety of today’s pie. Hence, we predict that the new equilibrium is war.

Mathematically, following the same procedure as in question 2, we can trivially deduce that the bargaining range is between $15 and $35 to Sparta. Hence, in order to avoid war, we need a pair \((x_1, x_2) \in BR'\), where

\[
BR' = \{(x_1, x_2) \in [-10, 10]^2 : x_1 + x_2 \geq 40\}
\]

But clearly, \(BR' = \{\emptyset\}\). Of course, those "40" come from the difference between the minimal amount that would leave Sparta indifferent by choosing war or peace, 140, and the shares that Sparta has in both periods, \(0.75 \times 100 + 0.25 \times 100\). As mentioned above, since the maximal amount of money Athens could compromise is $35, then war is the predicted equilibrium.

To prove that war is the resulting equilibrium, let’s assume in period 0 Athens offers as much as they could to try to make the offer as credible as possible. Then, if we show that even for \(x_1 = 25\) the equilibrium is war, then for any other feasible \(x_1\) the outcome would be the same.

**Period 0:** Athens, A, makes a take-it-or-leave-it offer \(x_1\). The two actions at this node are: \(x_1 = 25\) or \(x_1 \neq 25\). If A chooses the latter, the game ends in war where the expected payoffs are \((40, 140)\) for Athens and Sparta, respectively. If A chooses the former, the game continues.

**Period 1:** Sparta, S, accepts or rejects \(x_1\). If Spartans accept it the game continues. If they reject it the game ends in war where the expected payoffs are \((40, 140)\).

**Period 2:** Athens offers \(x_2\). Where the actions are \(x_2 = 40\) or \(x_2 < 40\). If \(x_2 = 40\) is chosen, the outcome is peace with \((60, 140)\) as the payoff. If \(x_2 < 40\) is chosen, we have war with \((25 - 25 + 65, 75 + 25 + 15) = (65, 115)\) as the outcome. Notice that in the latter payoff there was peace in period 1 and war occurs only in period 2. Thus, we add the expected value of war, $65 and $15 for Athens and Sparta, respectively, to the peace payoffs of period 1.

By backwards induction, we have two SPNE. In both of them the equilibrium is war. The first one is

\[
\{(x_1 = 25, x_2 < 40), \text{Reject}\}.
\]

The second one is

\[
\{(x_1 \neq 25, x_2 < 40), \text{Reject}\}.
\]