APMTH-105 Notes Section #1

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Goals for the week

(a) Learn some methods of constructing exact solutions for first-order ODEs.

(b) Learn how to find approximate solutions for ODEs using dominant balance.

(c) Using numerical approaches (MATLAB) to see how solutions behave.

Problem 1: Integrating Factors for linear equation

We will now work out a integrating factor problem in the way we discussed in Lecture 2. Consider the nonhomogeneous linear equation:

$$\frac{dy}{dt} + \frac{2}{t}y = t - 1 \tag{1}$$

Which is of the form,

$$\frac{dy}{dt} + g(t)y = b(t) \tag{2}$$

we first compute the integrating factor:

$$\sigma(t) = e^{\int g(t)dt} = e^{\int (2/t)dt} = e^{2\ln(t)} = e^{\ln(t^2)} = t^2 \tag{3}$$

Remember, the idea behind this method is to multiply both sides of the differential equation by $\sigma(t)$ so that the left-hand side of the new equation is the result of the product rule. In this case, multiplying by $\sigma(t) = t^2$ yields:

$$t^2\frac{dy}{dt} + 2ty = t^2(t-1) \tag{4}$$

Which we can rewrite as:

$$\frac{d}{dt}(t^2y) = t^3 - t^2 \tag{5}$$

Now integrating both sides with respect to t we get:

$$t^2y = \frac{t^4}{4} - \frac{t^3}{3} + c \tag{6}$$

Where C is an arbitrary constant. The general solution thus yields:

$$\boxed{y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{k}{t^2} \text{ where } k \text{ is found using a initial condition}} \tag{7}$$

Problem 2: Constructing exact solution for an initial value problem

Let's consider a first order differential equation of form

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)} \text{ with initial condition } y(0) = 2 \tag{8}$$

By re-writing the differential equation as

$$(\cos x \sin x - xy^2)dx + y(1-x^2)dy = 0 \tag{9}$$

Which is of the form:

$$M(x,y)dx + N(x,y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{10}$$

where:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial N}{\partial x} \tag{11}$$

we recognize that the equation is exact because

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x} \tag{12}$$

Now

$$\frac{\partial f}{\partial y} = y(1-x^2) \tag{13}$$

$$f(x,y) = \frac{y^2}{2}(1-x^2) + h(x) \tag{14}$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2 \tag{15}$$

The last implies that $h'(x) = \cos x \sin x$. Integrating this gives

$$h(x) = -\int (\cos x)(-\sin x dx) = -\frac{1}{2}\cos^2 x \tag{16}$$

Thus

$$\frac{y^2}{2}(1-x^2) - \frac{1}{2}\cos^2 x = c_1 \qquad \text{or} \qquad y^2(1-x^2) - \cos^2 x = c \tag{17}$$

where $2c_1$ had been replaced by c. The initial condition $y = 2$ when $x = 0$ demands that $4(1) - \cos^2(0) = c$, and so $c = 3$. Therefore an implicit solution is then

$$\boxed{y^2(1-x^2) - \cos^2 x = 3} \tag{18}$$

Problem 3: Constructing exact solution for problem using substitution

Consider the following differential equation:

$$\frac{dy}{dx} = \frac{-3y}{3x - 7y} \tag{19}$$

We recognize that the function on the left is homogeneous so we can use the substitution $y = ux$. By product rule we get that $\frac{dy}{dx} = x\frac{du}{dx} + u$ and making these substitutions we obtain

$$x\frac{du}{dx} + u = \frac{-3ux}{3x - 7ux} \tag{20}$$

Now we need to separate this equation such that all the u's are on one side and all the x's are on the other side. After some algebra we get that

$$\frac{3-7u}{-6u+7u^2}du = \frac{dx}{x} \tag{21}$$

Integrating this equation

$$\int \frac{3-7u}{-6u+7u^2}du = \int \frac{dx}{x} \tag{22}$$

we get

$$-\frac{1}{2}\ln(-6u+7u^2) = \ln(x) + C \tag{23}$$

$$\frac{1}{\sqrt{-6u+7u^2}} = c_1 x \Longrightarrow \frac{1}{-6u+7u^2} = c_1^2 x^2 = c_2 x^2 \tag{24}$$

Finally we use that $u = \frac{y}{x}$ to get our implicit solution

$$\frac{x^2}{-6yx+7y^2} = cx^2 \Longrightarrow \boxed{-6yx+7y^2 = c} \tag{25}$$

Problem 4: Constructing exact solution for problem using substitution, Bernoulli Equation

A Bernoulli Equation is of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{26}$$

Where we make substitutions for form $v = y^{1-n}$. Now consider the following equation which is a Bernoulli equation

$$\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3 \tag{27}$$

This is a Bernoulli equation with $n = 3$, $P(x) = -5$, $Q(x) = -5x/2$. To transform (27) into a linear eqation, we must first divide by y^3 to obtain:

$$y^{-3}\frac{dy}{dx} - 5y^{-2} = -\frac{5}{2}x \tag{28}$$

Next we make the substitution $v = y^{-2}$. Since $dv/dx = -2y^{-3}dy/dx$, the transformed equation is

$$-\frac{1}{2}\frac{dv}{dx} - 5v = -\frac{5}{2}x \tag{29}$$

$$\frac{dv}{dx} + 10v = 5x \tag{30}$$

Now we can see that equation (30) is linear, so we can solve it for v using the method of integrating factors for a linear first-order equations, where $g(x) = 10$, making

$$\sigma(x) = e^{\int 10dx} = e^{10x+C} = Ce^{10x} \tag{31}$$

$$\frac{dv}{dx}e^{10x} + 10ve^{10x} = 5xe^{10x} \tag{32}$$

$$\frac{d}{dx}(ve^{10x}) = 5xe^{10x} \tag{33}$$

$$\int \frac{d}{dx}(ve^{10x})dx = \int 5xe^{10x}dx \tag{34}$$

$$ve^{10x} = 5\int xe^{10x}dx \tag{35}$$

I know everybody hates this but we must pull out from the back of our minds that distant idea of 'Integration by parts', where $\int fdg = fg - \int gdf$, where we will let $f = x, dg = e^{10x}$, thus making $df = dx$ and $g = e^{10x}/10$. So that

$$5\int xe^{10x}dx = \frac{1}{2}e^{10x}x - \frac{1}{2}\int e^{10x}dx \tag{36}$$

For the integral of e^{10}, use substitution $u = 10x$ and $du = 10dx$ so that

$$5\int xe^{10x}dx = \frac{1}{2}e^{10x}x - \frac{1}{20}\int e^{u}du \tag{37}$$

Which simply becomes

$$\frac{1}{2}e^{10x}x - \frac{e^{u}}{20} + C \tag{38}$$

$$\frac{1}{2}e^{10x}x - \frac{e^{10x}}{20} + C \tag{39}$$

Plugging that back into our original equation (35)

$$ve^{10x} = \frac{1}{2}e^{10x}x - \frac{e^{10x}}{20} + C \tag{40}$$

$$v = \frac{1}{2}x - \frac{1}{20} + e^{-10x}C \tag{41}$$

Substituting back $v = y^{-2}$ we get a closed form solution for $y(x)$:

$$\boxed{y^{-2} = \frac{1}{2}x - \frac{1}{20} + e^{-10x}C} \tag{42}$$

Problem 5: Constructing approximate solutions using dominant balance

Consider the same Bernoulli equation (27) as in Problem 4. Now let's try to solve it using dominant balance for $x \to \infty$. This equation is of form $A + B + C + D = 0$, so we know we can use dominant balance.

$$\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3 \tag{43}$$

Since I know the answer, I am going to skip the few first dominant balances to save time and am going to go straight for the one that works and is consistent. So lets choose the second and last term such that we have

$$-5y + \frac{5}{2}xy^3 = 0 \tag{44}$$

$$1 = \frac{1}{2}xy^2 \tag{45}$$

Now solving for y here we will get:

$$\boxed{y = \left(\frac{2}{x}\right)^{1/2} \quad \text{for } x \to \infty} \tag{46}$$

In order to check for consistency we must compare the magnitude of the omitted terms to the magnitude of the kept terms. For this we have two options of kept terms and only one option for the omitted term. So I will choose the easier of the two kept terms which is $|5y|$ and will compare it to $|\frac{dy}{dx}|$. So from above we know that

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2^{1/2}}{x^{1/2}}\right) \approx \frac{1}{x^{3/2}} \tag{47}$$

Comparing the omitted and the kept term we see that the omitted term is much smaller than the kept term for $x \to \infty$

$$\frac{1}{x^{3/2}} << \frac{1}{x^{1/2}} \tag{48}$$

Making this balance consistent. So, now if we compare it to the solution found in Problem 4 equation (42) we can see that as $x \to \infty$ the terms e^{-10x} and $\frac{1}{20}$ become negligible compared to the other terms in the solution. This example shows us how asymptotic analysis can be a valuable time saver when only interested in a certain regime of values. I think everyone would agree that this method was much less painful than the one illustrated in Problem 4.

You should also check all other balances!

MATLAB Tutorial

Now let's look at solving the same equation as in Problem 4 and 5 numerically using MATLAB's built in ODE-45 solver.

$$\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3 \tag{49}$$

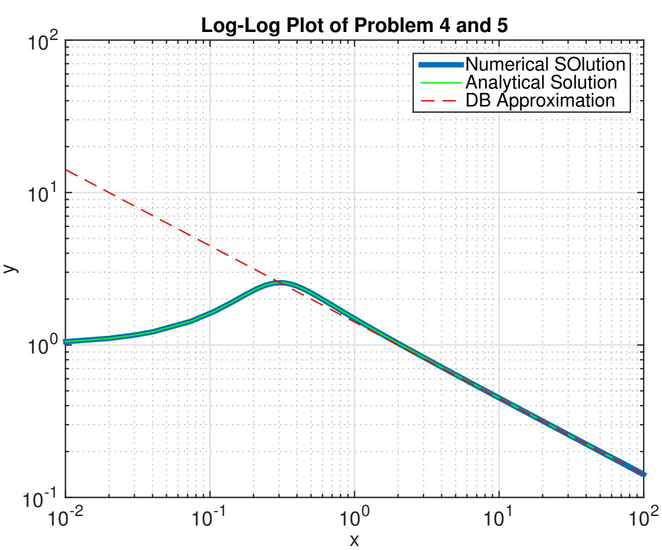

Figure 1: Graph of solutions for problems 4 and 5. Code for this graph can be seen below.

```
%Harvard Univeristy - AM105 Example
%By: Matheus Fernandes
clc
close all
clear all

dydx= @(x,y) 5*y-5/2*x*y^3;
[X,Y] = ode45(dydx,[0 100],1);


analytical= (2.*sqrt(5).*exp(5*X))./sqrt(exp(10.*X).*(10.*X-1)+21);

figure(1)
loglog(X,Y,'linewidth',4); hold on
loglog(X,analytical,'g')
loglog(X,(2./X).^(1/2),'r--')

title('Log-Log Plot of Problem 4 and 5','fontsize',15)
xlabel('x','fontsize',15)
ylabel('y','fontsize',15)
legend('Numerical SOlution','Analytical Solution','DB Approximation')
set(gca,'fontsize',15)
grid on
```