Testing for Interaction in Multiple Regression[1]

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Contrary to a recent claim, the inclusion of a product term in a multiple regression is a legitimate way to test for interaction. The unstandardized coefficient and the t-test for the product term are unaffected by the addition of arbitrary constants to the variables in the model. Certain other statistics are affected by this change, however, indicating that some hypotheses relating to interaction are not meaningfully testable unless variables are measured on ratio scales.

Sociological theories often imply that two or more variables interact in their effects upon some dependent variable (Blalock 1965). The variables x_1 and x_2 are said to interact in the determination of y if the effect of x_1 on y depends on the level of x_2 (which implies, symmetrically, that the effect of x_2 on y depends on the level of x_1). If all three variables are measured on numerical scales, it is common practice to test for the presence of interaction by including the product of x_1 and x_2 as an additional variable in a multiple regression. That is, one runs the model

$$\hat{y} = b_0 + b_1x_1 + b_2x_2 + b_3x_1x_2 \,. \tag{1}$$

If b_3 differs significantly from zero, the interaction is said to be significant.

Althauser (1971) has suggested that this method is invalid. He claims to show that the standardized coefficients corresponding to the b's in the above equation are strongly influenced by the correlation between either x_1 or x_2 and the product term x_1x_2. These correlations, in turn, depend on the sample means of x_1 and x_2. He concludes that the F-test for the coefficient b_3 is also affected by the sample means, a rather undesirable result since many sociological scales have arbitrary zero points and hence arbitrary means. This would seem to imply an arbitrary value for the test statistic as well.

I will argue here that, on the contrary, the inclusion of product terms in a multiple regression is a quite legitimate method for testing and estimating interaction effects.[2] While Althauser is correct in stating that the standard-

[1] After writing the initial draft of this paper, I learned that Arthur S. Goldberger had anticipated the central result in a personal communication with Robert Althauser. His formulation of the issue has helped to clarify my own. I am also grateful to George Bohrnstedt, Lowell Hargens, James House, and Scott Long for helpful suggestions.

[2] Because Althauser's approach is fundamentally different from the one taken here, I will not examine his arguments in detail. I only wish to note, first, that his statement about the F-ratio is not supported by a formal derivation. Second, his argument is marred by an error which occurs early in the paper and is carried through the remaining derivations.

ized coefficient for the product term is affected by changes in the means, the F- and t-ratios for the product term are not affected. Moreover, the unstandardized coefficient for the product term is also unaffected. Nevertheless, there are problems in the formulation and interpretation of tests for interaction when one or more of the variables are measured on interval scales. These difficulties will be explored in some detail.

CHANGING THE ZERO POINT OF AN INTERVAL SCALE

It is well known that the usual regression model assumes that the variables are measured at least by means of interval scales, a matter of some concern to those who have only ordinal data. But even those sociological variables which approximate an interval scale often have arbitrary zero points. Most attitude and prestige scales, for example, are of this type. Variables like income or population size which do have a theoretically fixed zero point are referred to as ratio scales (Hays 1963).

The distinction between interval and ratio scales is ordinarily of little importance in regression analysis. Most researchers are aware of the fact that adding a constant to a variable (which amounts to changing the zero point) changes the intercept in the equation but leaves the slope estimate unchanged. This creates little difficulty because the intercept is rarely an object for interpretation or testing.

Something very similar happens when a product term containing an interval variable is included in a regression equation. Consider equation (1), and suppose that x_1 is measured on an interval scale.[3] This means that the zero point of x_1 can be altered by adding or subtracting an arbitrary constant without any loss of information. Define $z_1 = x_1 + c$ where c is an arbitrary constant. Substituting into (1) yields

$$\hat{y} = b_0 + b_1(z_1 - c) + b_2x_2 + b_3(z_1 - c)x_2 \,. \tag{2}$$

Multiplying out and combining terms gives

$$\hat{y} = (b_0 - b_1c) + b_1z_1 + (b_2 - b_3c)x_2 + b_3z_1x_2 \,, \tag{3}$$

which can be rewritten as

$$\hat{y} = b_0{}^* + b_1z_1 + b_2{}^*x_2 + b_3z_1x_2 \tag{4}$$

where $b_0{}^* = b_0 - b_1c$ and $b_2{}^* = b_2 - b_3c$.

Equation (4) shows what would happen if y were regressed on z_1, x_2, and z_1x_2. In other words, it shows the consequences of arbitrarily changing the

Specifically, his equation (A4) purports to be a general expression for the variance of x_1x_2, but in fact it only holds when x_1 and x_2 are independent. Yet those same variables are assumed to be correlated in later sections of the paper.

[3] The results which follow are true for both the population and the sample. To simplify the discussion, they are formulated in terms of the sample statistics alone.

zero point of x_1, a permissible transformation if x_1 is measured at the interval level. We find that two coefficients are changed, the intercept and slope for x_2 alone. But the slopes for x_1 and the product term x_1x_2 are unaltered by this transformation. In a similar fashion, it is easy to show that if the regression is rerun with arbitrary constants added to both x_1 and x_2, all the coefficients *except* b_3 are changed.

This demonstrates that the coefficient for the product term is unaltered by changes in the means of the variables resulting from the addition of a constant to all scores. The invariance also extends to tests of hypotheses about b_3. For example, as Althauser (1971) points out, the hypothesis that $b_3 = 0$ in the population can be tested by forming the F-ratio

$$F = \frac{(R_B^2 - R_A^2)(N - 4)}{1 - R_B^2} \tag{5}$$

where R_B^2 is the coefficient of determination for (1), R_A^2 is for a model that excludes the product term, and N is the number of observations. It is well known that R_A^2 is unaffected by linear transformations of the independent variables in the regression. It is easily seen that the same invariance must also hold for R_B^2. Since only the right side of (4) is altered, the predicted value of y remains the same. This implies that R_B^2, which equals

$$\sum_{i=1}^{N} (y_i - \hat{y}_i)^2 \Big/ \sum_{i=1}^{N} (y_i - \bar{y})^2 ,$$

is unaltered by the transformation, and thus (5) is unchanged as well. Note that since the t-ratio for the hypothesis that $b_3 = 0$ is just the square root of (5), it also is unchanged.

The F-ratio for the coefficient of x_2, however, is changed by the addition of a constant to x_1 since $b_2 = 0$ and $b_2^* = 0$ are quite different hypotheses. In particular, if $b_2 = 0$ then b_2^* cannot equal zero unless there is no interaction (i.e., $b_3 = 0$) or unless $c = 0$ (a trivial case). Note also that by picking an appropriate value for c, one can make b_2^* take on any desired value. For instance, if $c = b_2/b_3$ then $b_2^* = 0$.

STANDARDIZED COEFFICIENTS

The preceding results for the unstandardized coefficients do not apply to the standardized (path) coefficients, which show a different pattern of change and invariance. The standardized coefficients for the model in (1) are given by

$$\begin{aligned} p_{y1} &= b_1 s_{x_1}/s_y \\ p_{y2} &= b_2 s_{x_2}/s_y \\ p_{y3} &= b_3 s_{(x_1x_2)}/s_y \end{aligned} \tag{6}$$

where the s's are standard deviations of the subscripted variables. In particular, $s_{(x_1x_2)}$ is the standard deviation of the product term. I have already shown that b_1 and b_3 are unchanged by the addition of an arbitrary constant to x_1, and obviously s_y, s_{x_2}, and s_{x_1} are also unchanged by that transformation. Therefore p_{y1} is not affected. The transformation does alter b_2, however, so p_{y2} changes with the addition of a constant to x_1. The coefficient p_{y3} depends on $s_{(x_1x_2)}$, which turns out to be affected by changes in the mean of x_1 or x_2. In the special case in which x_1 and x_2 are independent,

$$\sigma^2_{x_1x_2} = \mathrm{var}(x_1)\,\mathrm{var}(x_2) + E^2(x_1)\,\mathrm{var}(x_2) + E^2(x_2)\,\mathrm{var}(x_1)\,. \qquad (7)$$

This result (Goodman 1960) is for the population but it has an analogue for a sample. It shows that the variance of the product increases as the means of x_1 and x_2 depart from zero. In the general case in which x_1 and x_2 are not independent, the variance of the product is a somewhat more complicated function of the means (Bohrnstedt and Goldberger 1969). We find, therefore, that the only standardized coefficient not altered by a change in the zero point of one of the variables is the coefficient for that variable entered singly. And if the zero point changes for all the variables in the product term, then all the standardized coefficients will change.

AN EMPIRICAL EXAMPLE

The reader may easily demonstrate these results to himself by changing the coding in sample regressions with product terms. I present here a simple example using real data from a survey of parapsychologists.[4] For the 119 cases, there were measurements on three variables: x_1, x_2, and y. Panel A of table 1 shows results from an ordinary least-squares regression of y on x_1 and x_2. The two independent variables had positive effects on y that were both significant beyond the .01 level (one-tailed test). Panel B shows the regression of y on x_1 and x_2 with the addition of the product term x_1x_2. The coefficient for the product is significant at the .05 level (one-tailed test).

Panel C shows the results of a similar regression with x_1 recoded such that $x_1^* = x_1 + 400$. As predicted, the unstandardized coefficient for the product term and its associated t-ratio are unchanged. Also unchanged are the unstandardized coefficient, standardized coefficient, and t-ratio for x_1^*. But the intercept and the coefficient for x_2 both increase by two orders of magnitude. The t-ratio for x_2 doubles in magnitude, while the standardized coefficients for x_2 and $x_1^*x_2$ reach the enormous values of -34 and 34, re-

[4] The population consisted of 119 members of the Parapsychological Association. The three variables are x_1, a 12-point scale measuring interest in parapsychology; x_2, the number of professional associations in which they reported membership; y, the number of times they had experienced discrimination because of their interest in parapsychology. Strictly speaking, it is not permissible to add a constant to x_2 since it is measured as a ratio scale. This was done in panel D of table 1 for the sake of example.

TABLE 1

REGRESSIONS OF y ON x_1, x_2, AND x_1x_2, SHOWING EFFECTS OF ADDING CONSTANTS TO x_1 AND x_2

	b	Standardized Coefficient	t	R^2
		A		
Intercept	− 2.19	...	−2.34	.173
x_1	0.395	0.27	3.18	...
x_2	0.314	0.29	2.48	...
		B		
Intercept	− 0.456	...	−0.35	.199
x_1	0.152	0.10	0.87	...
x_2	− 0.336	− 0.31	−0.96	...
x_1x_2	0.0887	0.66	1.91	...
		C		
Intercept	−61.4	...	−0.86	.199
x_1*	0.152	0.10	0.87	...
x_2	−35.8	−33.56	−1.90	...
x_1*x_2	0.0887	33.87	1.91	...
		D		
Intercept	0.121	...	0.13	.199
x_1**	0.000	0.00	0.00	...
x_2	0.000	0.00	0.00	...
x_1**x_2*	0.0887	0.45	1.91	...

NOTE.—$x_1^* = x_1 + 400$, $x_1^{**} = x_1 - 3.788$, $x_2^* = x_2 + 1.714$.

spectively. Panel D shows results with both independent variables recoded as follows: $x_1^{**} = x_1 - 3.788$ and $x_2^* = x_2 + 1.714$. These values were derived from the results in panel B in order to produce the results shown here: unstandardized coefficients, standardized coefficients, and t-ratios for x_1** and x_2* all go to zero. But the unstandardized coefficient and t-ratio for the product term remain unchanged. Note also that R^2 is the same for all three models with product terms.

IMPLICATIONS

What do these results mean for the practicing researcher? First, although there may be better methods in some cases, the use of product terms to test for interaction is a legitimate method. The unstandardized coefficient and the t-test are not affected by changes in the means or zero points of the variables. Second, when one or more of the variables in the product term are measured on interval scales, it is useless to attempt to substantively interpret or test hypotheses about the coefficients for the other variables entered singly. If one of the variables has an arbitrary zero point, then those coefficients are also arbitrary. From a purely statistical point of view, one can validly test whether any of the coefficients differ from zero. The point is that since the magnitude of the coefficients depends on an arbitrary constant, there can be no theoretical basis for hypothesizing that the coefficients are zero. A hypothesis about the values of those coefficients is equivalent to a hypothesis fixing the zero point of the interval scale. In Fararo's (1973) terminology, such hypotheses are not "empirically meaningful statements." Third, for similar reasons it is an exercise in futility to attempt to determine the relative importance of main effects and interaction by examining the standardized coefficients. Even when variables are measured on ratio scales, this will probably not be an informative comparison. Perhaps the best measure of the importance of the interaction is simply the increment to R^2 with the inclusion of the product term. Finally, it must be emphasized that the transformations that have been considered do not alter the models in any fundamental respect—they merely rearrange the information that they contain. Nothing that is theoretically meaningful is lost by this transformation, but it is important to keep in mind which statistics convey theoretically useful information and which do not.

HIGHER-ORDER INTERACTIONS

All these results can be generalized in a straightforward manner to regression equations containing product terms with three or more factors. Consider the model

$$\hat{y} = b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_1x_2 + b_5x_1x_3 + b_6x_2x_3 + b_7x_1x_2x_3 \,. \tag{8}$$

Transforming this equation to one in $z_1 = x_1 + c$ gives

$$\hat{y} = b_0{}^* + b_1z_1 + b_2{}^*x_2 + b_3{}^*x_3 + b_4z_1x_2 + b_5z_1x_3 + b_6{}^*x_2x_3 + b_7z_1x_2x_3 \tag{9}$$

where an asterisk indicates that the coefficient has changed with the transformation. For equations with all possible interactions and main effects, the general rule is this: Coefficients for all terms involving the transformed (interval) variable are unaltered; coefficients for all other terms are changed (arbitrary). More complicated rules are needed when some of the possible terms are excluded. For example, if b_7 in (8) is set equal to zero, the transformation of x_1 will affect only b_0, b_2, and b_3, leaving b_1, b_4, b_5, and b_6 unchanged.

HIERARCHICAL TESTING

It is a common rule of thumb that testing for interaction in multiple regression should only be done hierarchically. That is, one should test for higher-order interactions only when all lower-order interactions and main effects are included in the equation. If a rationale for this rule is given at all, it is usually that additive relationships somehow have priority over multiplicative relationships. When all variables are measured at the ratio level and there are strong theoretical reasons for excluding lower-order terms, this rule seems overly stringent. Indeed, the exclusion of lower-order terms in such cases can increase precision of estimation and the power of hypothesis tests (Wonnacott and Wonnacott 1970, p. 312). But when one or more variables in a product term are measured at the interval level, the hierarchical principle becomes essential.

Suppose, for example, that x_1, x_2, and y are measured at the ratio level and there exists a theory which states that the appropriate model is

$$\hat{y} = bx_1x_2 \,. \tag{10}$$

In this case it would be quite reasonable to run the regression corresponding to (10) in order to estimate b. But suppose that x_1 and x_2 cannot be measured directly and one has to settle for the interval measures $z_1 = x_1 + c$ and $z_2 = x_2 + d$ where c and d are unknown constants. Substituting these measures into (10) yields

$$\begin{aligned} \hat{y} &= b(z_1 - c)(z_2 - d) \\ &= bcd - bdz_1 - bcz_2 + bz_1z_2 \\ &= a_0 + a_1z_1 + a_2z_2 + a_3z_1z_2 \end{aligned} \tag{11}$$

where the a_i coefficients are defined by the line preceding them. This result indicates that, although the original hypothesis in (10) sets the lower-order terms equal to zero, that fact does not imply that the lower-order terms can be excluded when running the regression on interval-level variables. It should be clear that a_0, a_1, and a_2 can be zero or nonzero depending on the unknown values of c and d. Hence, there can be no theoretical justification for setting these coefficients equal to zero (i.e., excluding the terms from the equation). Moreover, if the lower-order terms are excluded, the coefficient for the product term and the R^2 for the model will vary with the unknown values of c and d. Thus, the earlier conclusions about invariance are true only when all lower-order terms are included.[5]

OTHER MODELS FOR INTERACTION

This line of reasoning leads to the conclusion that many potential models for interaction cannot be estimated when some of the variables are measured

[5] The model in (10) does impose a constraint on the a_i coefficients in the last line of equation (11). Specifically, it requires that $a_0a_3 = a_1a_2$. Testing and estimating the model under this constraint requires nonlinear regression techniques of the sort discussed by Draper and Smith (1966).

on interval scales. For example, consider

$$\hat{y} = b_0 x_1^{b_1} x_2^{b_2}\,, \tag{12}$$

of which (10) is a special case.[6] A frequently suggested approach to estimating this model (Wonnacott and Wonnacott 1970) is to take the logarithm of both sides to produce

$$\log \hat{y} = \log b_0 + b_1 \log x_1 + b_2 \log x_2 \tag{13}$$

and simply regress log y on log x_1 and log x_2. But notice that if $z_1 = x_1 + c$ is substituted into (13), there is no way to simplify so that c is absorbed into one of the other parameters. Since c becomes an additional unknown parameter, the model is underidentified and cannot be estimated. Again, if one goes ahead and regresses log y on log z_1 and log x_2, the coefficients and the R^2 will vary with c.

The same argument applies to ratios of scores when the denominator can only be measured at the interval level. Suppose the theoretical model is

$$\hat{y} = b_0 + b_1 x_1 + b_2 x_2 + b_3(x_2/x_1)\,. \tag{14}$$

If one has only z_1 instead of x_1, the model cannot be validly estimated; the results will differ for every value of c. If, on the other hand, one has a ratio-level measure of x_1 but only an interval measure of x_2, the model can be transformed so that it is not affected by the addition of a constant to x_2:

$$\hat{y} = b_0{}^* + b_1 x_1 + b_2 z_2 + b_3(z_2/x_1) + b_4/x_1 \tag{15}$$

where $z_2 = x_2 + d$. In this case, d is absorbed into the parameters $b_0{}^*$ and b_4 and will not affect estimates of the other coefficients.

In general, it appears that when all independent variables are measured at the interval level the only valid way to test for interaction in the framework of least-squares regression is to include product terms and all lower-order terms in the equation. One can check whether any particular model is invariant to scale transformations of the variables by (*a*) writing the model as though all the variables were measured at the ratio level and (*b*) substituting variables which add arbitrary constants to the ratio variables. If the constants can be absorbed into one or more parameters, the model is invariant in the sense of generating the same predicted values for y and, hence, the same R^2.

A quite different approach is to reduce the interval-level variables to sets of categories and test for interaction by analysis of variance, analysis of covariance, or the equivalent multiple regression using dummy variables. Consider, first, the case of one dichotomous variable and one continuous

[6] This model is well known to economists as a Cobb-Douglas production function which relates output y to labor x_1 and capital x_2 under the constraint that $b_2 = 1 - b_1$ (Wonnacott and Wonnacott 1970).

variable. Suppose that y, x_1, and x_2 are all measured on interval scales and the model is

$$\hat{y} = b_0 + b_1x_1 + b_2w + b_3x_1w \tag{16}$$

where $w = s$ if $x_2 > m$ and $w = t$ if $x_2 \leq m$ and $s \neq t$. Since w is a nominal variable, s and t could have any values so long as $s \neq t$. It can be shown that recoding w to change the values of s and t could change any or all of the coefficients in the model. Yet, there is a sense in which the model remains unaffected by such changes. First, the R^2 will not change; second, the t-ratio for the hypothesis that $b_3 = 0$ will be unaltered. Thus, the test for the presence of interaction does not depend on recoding the variables in the product term.

By appropriate choice of s and t, moreover, one can get useful information from the coefficient estimates. The most common coding is to let $s = 1$ and $t = 0$ (or vice versa) which produces the following interpretation:

$$\hat{y} = b_0 + b_1x_1 \tag{17}$$

is the regression of y on x_1 for only those cases where $x_2 \leq m$;

$$\hat{y} = (b_0 + b_2) + (b_1 + b_3)x_1 \tag{18}$$

is the result that would be obtained if y were regressed on x_1 only for those cases where $x_2 > m$. Thus, b_3 is the difference between the slopes for the two groups, and b_2 is the difference in the intercepts.

This well-known result is discussed at length by Gujarati (1970*a*, 1970*b*), who also considers the case in which x_2 is divided into more than two categories. The important points in this context are that (*a*) *any* coding of w that maintains $s \neq t$ will give an invariant test for the presence of interaction; (*b*) if x_1 is measured on an interval scale, b_2 can be made equal to zero by adding an appropriate constant to x_1; (*c*) the standardized coefficients are essentially arbitrary; even with the one-zero coding they have no obvious interpretation.

The case in which both independent variables are reduced to sets of two or more categories is equivalent to two-way analysis of variance with disproportionate cell frequencies. The literature on this topic is too enormous to summarize here. For an introduction see Burke and Schuessler (1974), who point out the nonadditivity of sums of squares and the sensitivity of main effects to the arbitrary constraints imposed on the interaction effects.

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