ES120 Spring 2018 – Midterm 2 Solutions

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Length: 53 minutes

You are allowed to use a calculator when solving the problems, as well as the equation sheet posted on the web site. Please make sure your answers are clear and legible. No credit will be given if we cannot read an answer or figure out how you derived it! All questions are weighted equally.

Problem 1:

Consider a prismatic beam of length L, moment of inertia I, and Young's modulus E, simply supported on either end. The beam supports several discrete loads P (all equal) as shown in the picture. Use singularity functions to derive an expression for the deflection of the beam.

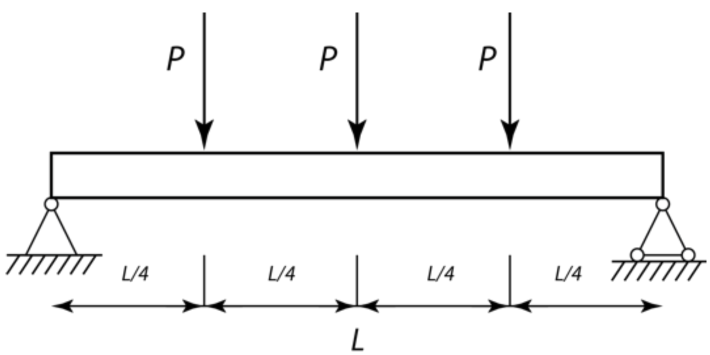


Figure 1: Problem 1 Schematic

Solution 1

To solve this problem, you can do force balance in y direction to yield:

$$\Sigma F_y = 0 \Rightarrow R_a - 3P + R_b = 0 \tag{1}$$

We can also note that the weight is distributed symmetrically so the $3P$ has to be allocated half to R_a and half to R_b such that:

$$R_a = \frac{3}{2}P \tag{2}$$

$$R_b = \frac{3}{2}P \tag{3}$$

Using this information we can come up with an equation for the shear force namely

$$V(x) = P\frac{3}{2} - P\left\langle x - \frac{L}{4}\right\rangle^0 - P\left\langle x - \frac{L}{2}\right\rangle^0 - P\left\langle x - \frac{3L}{4}\right\rangle^0 \tag{4}$$

By integrating we obtain $M(x)$ as

$$M(x) = Px\frac{3}{2} - P\left\langle x - \frac{L}{4}\right\rangle^1 - P\left\langle x - \frac{L}{2}\right\rangle^1 - P\left\langle x - \frac{3L}{4}\right\rangle^1 \tag{5}$$

Using the equation:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{6}$$

we obtain

$$EI\frac{d^2y}{dx^2} = Px\frac{3}{2} - P\left\langle x - \frac{L}{4}\right\rangle^1 - P\left\langle x - \frac{L}{2}\right\rangle^1 - P\left\langle x - \frac{3L}{4}\right\rangle^1 \tag{7}$$

which if we integrate we obtain

$$EI\frac{dy}{dx} = Px^2\frac{3}{4} - \frac{P}{2}\left\langle x - \frac{L}{4}\right\rangle^2 - \frac{P}{2}\left\langle x - \frac{L}{2}\right\rangle^2 - \frac{P}{2}\left\langle x - \frac{3L}{4}\right\rangle^2 + c_1 \tag{8}$$

Finally, if we integrate one last time, we obtain

$$EIy(x) = Px^3\frac{1}{4} - \frac{P}{6}\left\langle x - \frac{L}{4}\right\rangle^3 - \frac{P}{6}\left\langle x - \frac{L}{2}\right\rangle^3 - \frac{P}{6}\left\langle x - \frac{3L}{4}\right\rangle^3 + c_1x + c_2 \tag{9}$$

Here we note that we need 3 different boundary conditions to be able to solve for these constants. For this, the boundary conditions are at the left edge are

$$y(0) = 0 \text{ and } y''(0) = 0 \tag{10}$$

and at the right edge it is

$$y(L) = 0 \text{ and } y''(L) = 0. \tag{11}$$

Using the the following boundary conditions we obtain

$$y(0) = 0 \Rightarrow C_2 = 0 \tag{12}$$

$$y(L) = 0 \Rightarrow C_1 = -\frac{PL^2}{6}\left(\frac{3}{2} - \left(\frac{3}{4}\right)^3 - \left(\frac{1}{2}\right)^3 - \left(\frac{1}{4}\right)^3\right) = -\frac{5PL^2}{32} \tag{13}$$

So the final equation for the deflection becomes

$$\boxed{EIy(x) = Px^3\frac{1}{4} - x\frac{5PL^2}{32} - \frac{P}{6}\left\langle x - \frac{L}{4}\right\rangle^3 - \frac{P}{6}\left\langle x - \frac{L}{2}\right\rangle^3 - \frac{P}{6}\left\langle x - \frac{3L}{4}\right\rangle^3} \tag{14}$$

Note, that we have one final boundary condition, which we we need to satisfy, or at least verify that it is met. Namely that there is also no moment on the right side $y''(L) = 0$ If we plug L into the equation for $M(x)$ we can check that

$$M(L) = EI\frac{d^2y(x=L)}{dx^2} = PL\frac{3}{2} - P\left\langle L - \frac{L}{4}\right\rangle^1 - P\left\langle L - \frac{L}{2}\right\rangle^1 - P\left\langle L - \frac{3L}{4}\right\rangle^1 = 0 \tag{15}$$

and we obtain

$$M(L) = EI\frac{d^2y(x=L)}{dx^2} = PL\frac{3}{2} - PL\frac{3}{2} = 0 \checkmark \tag{16}$$

We can do the same thing for the boundary condition $y(L) = 0$ and verify that it is correct.

The final solution can be plotted using Matlab and normalizing $X^* = \frac{X}{L}$ and $P^* = \frac{P}{EI}$ to obtain the following plot

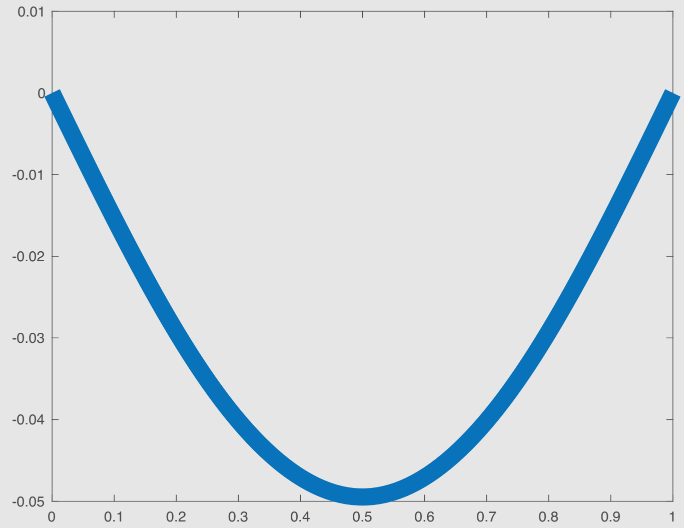


Figure 2: Deformation plot for beam in problem 1.

Matlab Code for Plotting Beam Deformation

```
function Plots()

X=linspace(0,1);

plot(X,P1(X,1,1),'Linewidth',12)
set(gcf, 'Color', 'None')

end


function val = P1(X,L,P)
ct=0;
for x = X
    ct=ct+1;
    val(ct) = P*x^3/4-5*P*L^2/32*x;

    if x>=L/4
        val(ct)=val(ct)-P/6*(x-L/4)^3;
    end

    if x>=L/2
        val(ct)=val(ct)-P/6*(x-L/2)^3;
    end

    if x>=3*L/4
        val(ct)=val(ct)-P/6*(x-3*L/4)^3;
    end
end


end
```

Note: The plot is not necessary for full credit and is just to illustrate the deformation visually.

Problem 2:

Consider a prismatic beam of length L, moment of inertia I, and Young's modulus E, clamped on the left. The beam is also supposed to rest on a support on the right side, but because of faulty construction there is a small gap δ. The beam is loaded by a uniform distributed load w_o (pointing down). (1) Determine the load wo at which the tip of the beam touches the support. (2) Calculate the force exerted on the support once the beam touches.

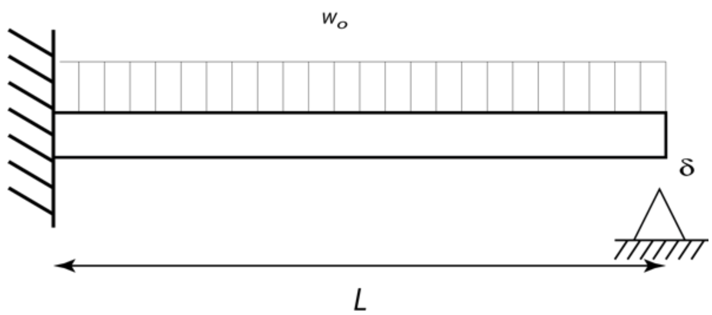


Figure 3: Problem 2 Schematic

Solution 2

For this problem we can simply consult Appendix D. Using appendix D case 2, we can get the following equation for the deflection

$$y(x) = -\frac{w_o}{24EI}\left(x^4 - 4Lx^3 + 6L^2x^2\right) \tag{17}$$

Part (1)

For this part we know that we need to match the deflection at the end of the beam to the gap δ, namely

$$y(x) = -\frac{w_o}{24EI}\left(L^4 - 4L^4 + 6L^4\right) = -\delta \tag{18}$$

Solving for w_o we obtain:

$$\boxed{w_o = \frac{8\delta EI}{L^4}} \tag{19}$$

Part (2)

For this part we now need to assume that we have case 1 of Appendix D. We will call the deflection resulting from the reaction force y^*. So the deflection equation here is

$$y^*(x) = -\frac{P}{6EI}\left(x^3 - 3Lx^2\right) \tag{20}$$

We know that on the right side of the beam we have the compatibility equation:

$$-y(L) + y^*(L) = \delta \Rightarrow y^*(L) = \delta + y(L) \tag{21}$$

Such that

$$y^*(L) = \delta - \frac{w_oL^4}{8EI} \tag{22}$$

Plugging in the left side of the equation

$$y^*(L) = -\frac{P}{6EI}\left(L^3 - 3L^3\right) = \delta - \frac{w_o L^4}{8EI} \tag{23}$$

Which we can solve for P as

$$\boxed{P = \frac{3\delta EI}{L^3} - \frac{3w_o L}{8} \quad \Big\downarrow \quad \text{Pointing down}} \tag{24}$$

Problem 3:

In a sailboat with a fractional rig, the tension in the backstay (i.e., the steel cable that runs from the masthead to the stern) is often used to bend the mast and thus shape the sails for optimum sailing performance in a race (bending the mast tends to depower the sails). As the wind increases, sailors increase the tension in the backstay using a system of pulleys or sometimes hydraulics. The figure below shows the geometry. Assume the mast can be modeled as a vertical beam that is simply supported at the deck level and at the point where the forestay (i.e., cable running from bow to mast) is connected to the mast. Assume further that the mast is initially straight. The bending stiffness of the mast is EI. The height of the mast is H; the height of the point where the forestay is connected is h. The distance from the bow to the mast is a, and from the mast to the stern is b. Determine the shape of the mast as a function of the tension S in the backstay. No need to consider buckling here. You do not need to account for any effects of wind in this problem.

Note that we assume the cables not to deform so that $Y(h) = 0$.

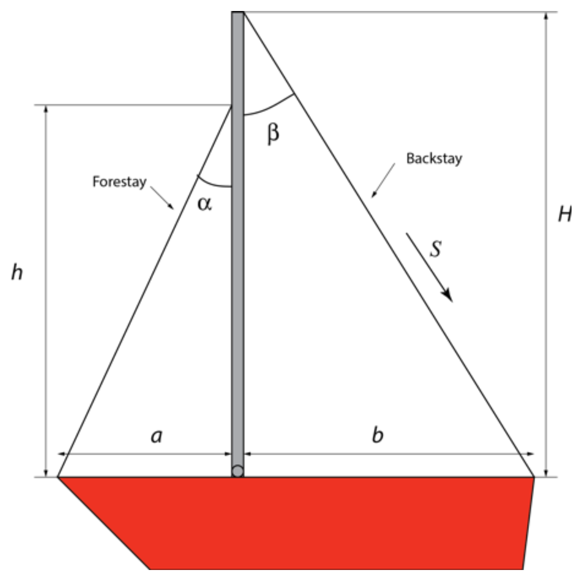


Figure 4: Problem 3 Schematic

Solution 3

First let's draw a free body diagram of the mast with the different tension components of the forestay and the backstay. We note that the attachment of the mast to the hull is a pin joint, based on the picture (no moments).

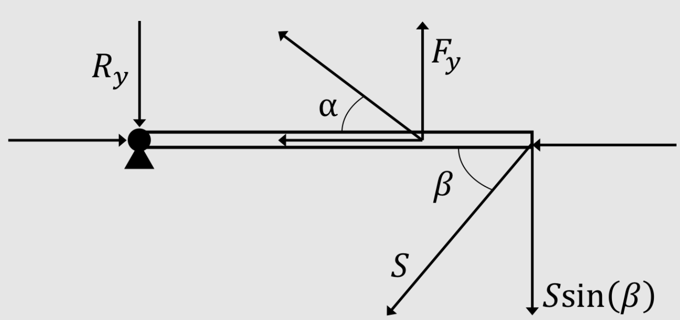


Figure 5: Free body diagram for depicted problem.

For this we need to begin by finding the reaction forces of the Forestay F_y and R_y. To do this, do sum of forces in the Y direction and sum of moments at the joint between the mast and the hull, respectively:

$$\Sigma F_y = 0: \quad -S\cos(\beta) + R_f - F_y \tag{25}$$

$$\Sigma M = 0: \quad -F_y(h) + S\sin(\beta)(H) = 0 \Rightarrow F_y = S\sin(\beta)\left(\frac{H}{h}\right) \tag{26}$$

Thus, using eq. (25),

$$R_y = S\sin(\beta)\left(\frac{H}{h} - 1\right) \quad \text{which is positive given that } \frac{H}{h} > 1 \tag{27}$$

Using these results we are ready to obtain the equations for the shear force, namely,

$$V(x) = -R_y + F_y < x - h >^0 \tag{28}$$

We can now integrate this to obtain the moment

$$M(x) = -R_y x + F_y < x - h >^1 \tag{29}$$

Again, knowing that

$$EI\frac{d^2y}{dx^2} = M(x) = -R_y x + F_y < x - h >^1 \tag{30}$$

We can integrate to obtain

$$EI\frac{dy}{dx} = -\frac{R_y}{2}x^2 + \frac{F_y}{2} < x - h >^2 + c_1 \tag{31}$$

and integrate again to obtain

$$EIy(x) = -\frac{R_y}{6}x^3 + \frac{F_y}{6} < x - h >^3 + c_1 x + c_2 \tag{32}$$

Here we use two boundary conditions, namely,

$$y(0) = 0 \quad \text{and} \quad y(h) = 0. \tag{33}$$

Using the first BC, we can solve

$$y(0) = 0 \Rightarrow c_2 = 0 \tag{34}$$

Using the second BC, we obtain

$$y(h) = -\frac{R_y}{6}h^3 + \frac{f_y}{6}(0) + c_1 h = 0 \Rightarrow c_1 = \frac{R_y}{6}h^2 \tag{35}$$

Putting all of this together, we obtain the final expression for the shape of the mast

$$EIy(x) = -\frac{R_y}{6}x^3 + \frac{F_y}{6} < x - h >^3 + \frac{R_y h^2}{6}x \tag{36}$$

$$y(x) = \frac{R_y}{EI6}(xh^2 - x^3) + \frac{F_y}{EI6} < x - h >^3 \tag{37}$$

Which if we plug in what we know for R_y and F_y, we can obtain an expression as a function of S

$$\boxed{y(x) = \frac{S\sin(\beta)\left(\frac{H}{h} - 1\right)}{EI6}(xh^2 - x^3) + \frac{S\sin(\beta)\left(\frac{H}{h}\right)}{EI6} < x - h >^3} \tag{38}$$

Which can also be written in terms of b instead of β (from simple geometry, $\sin(\beta) = \frac{b}{\sqrt{H^2+b^2}}$) is

$$y(x) = \frac{Sb\left(\frac{H}{h}-1\right)}{\sqrt{H^2+b^2}EI6}(xh^2-x^3) + \frac{Sb\left(\frac{H}{h}\right)}{\sqrt{H^2+b^2}EI6} < x-h >^3 \tag{39}$$

Note that this solution is only valid for $h < H$.

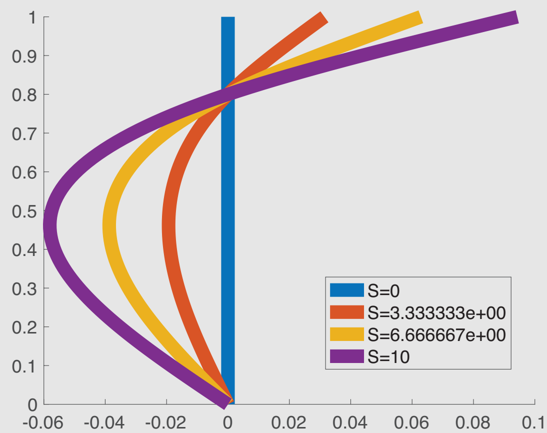


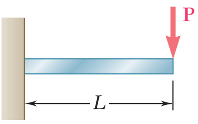
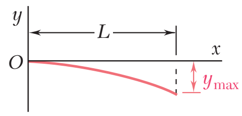
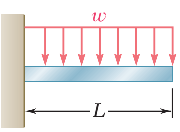
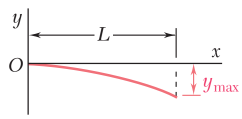
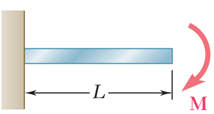
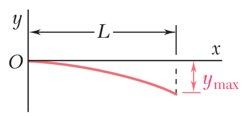
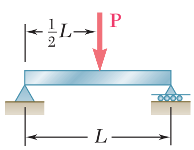
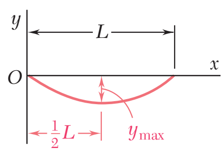
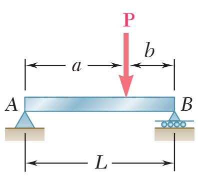
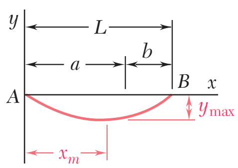
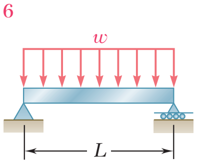
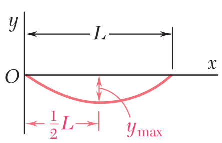
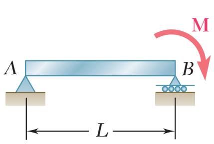
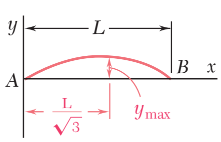
Figure 6: Deformation plot for beam in problem .

Matlab Code for Plotting Beam Deformation

```
function Plots2()

X=linspace(0,1);

S=linspace(0,10,4);
hold on
for s =S
plot(-P1(X,1,0.8,1,s,1,1),X,'Linewidth',12)
end
legend(cellstr(num2str(S', 'S=%-d')),'fontsize',16)
set(gca, 'fontsize', 16)

end


function val = P1(X,H,h,b,s,E,I)
ct=0;
for x = X
    ct=ct+1;
    val(ct) = (H/h-1)*(x*h^2-x^3);

    if x>=h
        val(ct)=val(ct)+(H/h)*(x-h)^3;
    end
    val(ct)=val(ct)*s*b/(sqrt(H^2+b^2)*E*I*6);

end


end
```

ES 120 -- Introduction to the Mechanics of Solids

APPENDIX D Beam Deflections and Slopes

Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
1 		$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y = \frac{P}{6EI}(x^3 - 3Lx^2)$
2 		$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y = -\frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
3 		$-\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y = -\frac{M}{2EI}x^2$
4 		$-\frac{PL^3}{48EI}$	$\pm\frac{PL^2}{16EI}$	For $x \le \frac{1}{2}L$: $y = \frac{P}{48EI}(4x^3 - 3L^2x)$
5 		For $a > b$: $-\frac{Pb(L^2-b^2)^{3/2}}{9\sqrt{3}EIL}$ at $x_m = \sqrt{\frac{L^2-b^2}{3}}$	$\theta_A = -\frac{Pb(L^2-b^2)}{6EIL}$ $\theta_B = +\frac{Pa(L^2-a^2)}{6EIL}$	For $x < a$: $y = \frac{Pb}{6EIL}[x^3 - (L^2-b^2)x]$ For $x = a$: $y = -\frac{Pa^2b^2}{3EIL}$
6 		$-\frac{5wL^4}{384EI}$	$\pm\frac{wL^3}{24EI}$	$y = -\frac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$
7 		$\frac{ML^2}{9\sqrt{3}EI}$	$\theta_A = +\frac{ML}{6EI}$ $\theta_B = -\frac{ML}{3EI}$	$y = -\frac{M}{6EIL}(x^3 - L^2x)$