ES120 Spring 2018 – Midterm 1 Solutions

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Length: 53 minutes

You are allowed to use a calculator when solving the problems, as well as the equation sheet posted on the web site. Please make sure your answers are clear and legible. No credit will be given if we cannot read an answer or figure out how you derived it! All questions are weighted equally.

Problem 1:

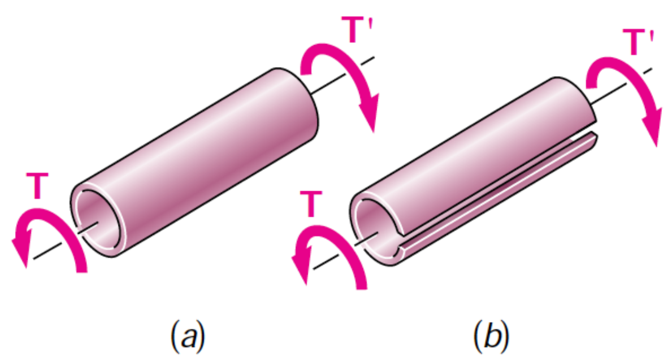

Figure 1

Equal torques are applied to thin-walled tubes of the same length L, same thickness t, and same radius c. One of the tubes has been slit lengthwise as shown. Determine (a) the ratio τ_b/τ_a of the maximum shearing stresses in the tubes, (b) the ratio ϕ_b/ϕ_a of the angles of twist of the shafts, (c) the radii of solid cylindrical shafts of the same material with the same stiffness as the two thin-walled tubes.

Solution 1

For this problem lets separate into separate geometries, namely 'a' for the one without the slit and 'b' for the one with the slit. This is not part a and part b of the problem, but simply a step towards the goals of the problem.

Here we are only calculating the different values for the different geometries. So for the geometry without the slit we can compute the following maximum shear stresses using the fact that we know this to be a thin-walled hollow shaft, which can be read in section 3.13 of the book. We know that the torque applied to a hollow member with constant shear flow is

$$\tau_a = \frac{T}{2t\mathcal{A}} \tag{1}$$

where $\mathcal{A}$ is the area bounded by the centerline of the wall cross section. Which given that t is very thin, we can approximate it to be simply c which gives us $\mathcal{A} = \pi c^2$. Therefore,

$$\tau_a = \frac{T}{2\pi c^2 t} \tag{2}$$

$$J \approx 2\pi c^3 t \tag{3}$$

So therefore, we can solve for the angle of twist as

$$\phi_a = \frac{TL}{GJ} = \frac{TL}{2\pi c^3 t G} \tag{4}$$

Now for the case for the geometry with the slit, we can approximate it as a non-circular member as described in section 3.12 of the book. For this approximation we can compute the length of the wider face a to be the circumference of the cylinder

$$a = 2\pi c \tag{5}$$

and the length of the lesser wide face to be

$$b = t. \tag{6}$$

We can see that

$$\frac{a}{b} = \frac{2\pi c}{t} >> 1 \tag{7}$$

which from Table 3.1 of the book indicates that $c_1 = c_2 = \frac{1}{3}$. From the equation in section 3.12 we can therefore compute the τ_b to be

$$\tau_b = \frac{T}{c_1 a b^2} = \frac{3T}{2\pi c t^2} \tag{8}$$

Now for the angle of twist we also know form section 3.12 of the book that it is given by:

$$\phi_b = \frac{TL}{c_2 a b^3 G} = \frac{3TL}{2\pi c t^3 G} \tag{9}$$

Part (a)

Now solving for the question on the stress ratio we can plug in our findings to get:

$$\frac{\tau_b}{\tau_a} = \frac{3T}{2\pi c t^2} \cdot \frac{2\pi c^2 t}{T} = \boxed{\frac{3c}{t}} \tag{10}$$

Part (b)

Now solving for the question on the twist ratio we can plug in our findings to get:

$$\frac{\phi_b}{\phi_a} = \frac{3TL}{2\pi c t^3 G} \cdot \frac{2\pi c^3 t G}{TL} = \boxed{\frac{3c^2}{t^2}} \tag{11}$$

Now let's look at these solutions a little deeper for some intuition. It is cool to see that when we add a slit to the thin shell cylinder we get higher maximum shear stresses than without the slit (assuming $c > t$)). Furthermore, not only is our angle of twist also larger for the geometry with the slit but it scales $\left(\frac{c}{t}\right)^2$ which is much faster than the maximum shear stresses.

Part (c)

For this part we expect two solutions for the radii given that there are two different tubes of different stiffnesses. To obtain a similar stiffness between two members, we must equate the relationship between the torque required per unit twist, namely ϕ.

Therefore, the torsional stiffness of a solid cylindrical shaft is

$$\phi_{\text{cyl}} = \frac{TL}{G\frac{\pi}{2}r_{\text{cyl}}^4} \tag{12}$$

For shaft (a) we must equate the two equations for torsional stiffness, namely

$$\phi_a = \frac{TL}{2\pi c^3 tG} = \frac{TL}{G\frac{\pi}{2}r_{\text{cyl}}^4} = \phi_{\text{cyl}} \tag{13}$$

$$\boxed{r_{\text{cyl}}^{(\text{a})} = \sqrt[4]{4c^3 t}} \tag{14}$$

For shaft (b) we must equate the two equations for torsional stiffness, namely

$$\phi_b = \frac{3TL}{2\pi ct^3 G} = \frac{TL}{G\frac{\pi}{2}r_{\text{cyl}}^4} = \phi_{\text{cyl}} \tag{15}$$

$$\boxed{r_{\text{cyl}}^{(\text{b})} = \sqrt[4]{\frac{4}{3}ct^3}} \tag{16}$$

Note, for this part we were very lenient if you attempted and got something remotely close. However, you had to identify that there are two radii that you are solving for in this problem.

Problem 2:

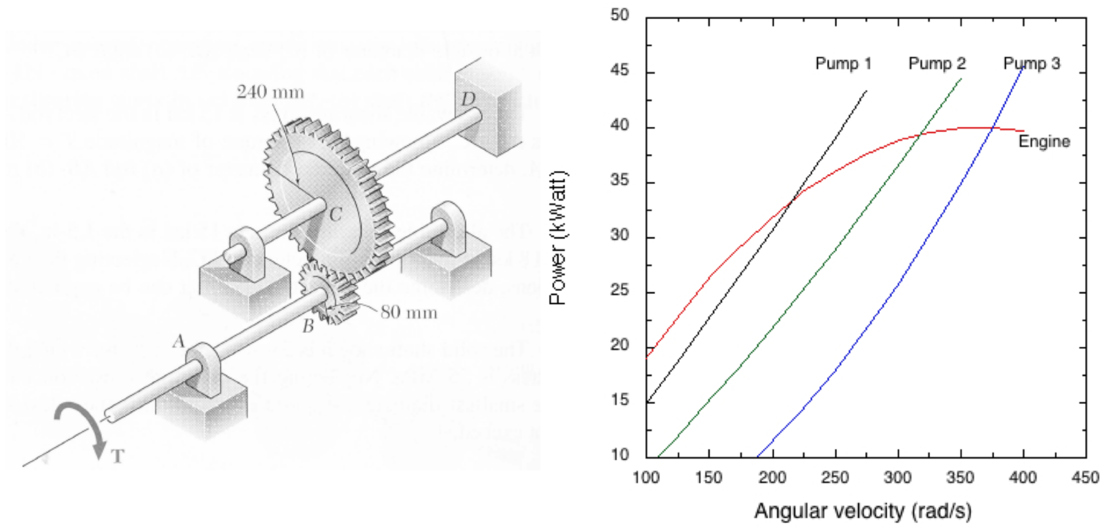

Figure 2

Consider the figure below and imagine that A represents a diesel engine. Gears B and C represent a gearbox, and D represents the load, in this case a pump. The maximum output of the engine is 40 kW. The characteristics of the engine (as measured at point D for a fixed fuel supply) and of several pumps are shown in the graph below.

(a) Given that you want to maximize power to the pump, which pump would you select?

(b) Having selected the pump, determine the corresponding angular velocities at points A and D. What are the internal torques in sections AB and CD of the shaft. What is the total elastic twist angle ϕ in the system as a result of these torques?

(c) Knowing that the maximum allowable shear stress of the shaft material (mild steel) is 105 MPa, determine the required diameter of shaft AB and shaft CD. Use metric units in your answer.

Solution 2

Part (a)

If we want to maximize power we can see by the intersection of the graph for the different pumps and the engine that the maximum power occurs when pump 3 intersects with the engine. Therefore, we would want to select **Pump 3**.

Part (b)

From the graph provided in the problem, we can see that the intersection occurs at a power $P = 40,000$ kW and $\omega = 375$ rad/s (we accept anywhere from 350-400 rad/s). Because the engine is acting at point A, we know that $\omega_{AB} = 375$ rad/s.

Using the gear ratio we can develop the following relationship between the angular momentum of the two shafts

$$\omega_{CD} = \frac{80}{240}\omega_{AB} \Rightarrow \omega_{CD} = 125 \text{ [rad/s]} \tag{17}$$

To obtain the torque, we can use the relationship between power and angular velocity,

$$T = \frac{p}{\omega} \tag{18}$$

Note, that even if you didn't remember this equation or could find it in the equation sheet, it would have been easy to come up with using dimensional analysis.

Now we can compute the different torques, namely,

$$\boxed{T_{AB} = \frac{40000}{375} = 106.66 \text{ N} \cdot \text{m}} \tag{19}$$

$$\boxed{T_{CD} = \frac{40000}{125} = 320 \text{ N} \cdot \text{m}} \tag{20}$$

Similarly, to compute the angle of twist we have the equation relating torque and angle of twist:

$$\phi = \frac{TL}{GI_p} \tag{21}$$

Where the polar moment of inertia for this cross section is

$$I_p = \frac{\pi}{2} r^4 \tag{22}$$

Plugging this it, it becomes

$$\phi = \frac{TL}{G\frac{\pi}{2}r^4} \tag{23}$$

Now we can compute the different twist angles for the different members, namely,

$$\phi_{AB} = \frac{T_{AB}L_{AB}}{G\frac{\pi}{2}r_{AB}^4} \tag{24}$$

$$\phi_{CD} = \frac{T_{CD}L_{CD}}{G\frac{\pi}{2}r_{CD}^4} \tag{25}$$

Using the information we were provided this far in the problem, we can come up with the total angle of twist to be

$$\boxed{\phi_{\text{total}} = \phi_{AB} + \phi_{CD} = \frac{106.66L_{AB}}{G\frac{\pi}{2}r_{AB}^4} + \frac{320L_{CD}}{G\frac{\pi}{2}r_{CD}^4}} \tag{26}$$

Note, we did not take points off for incorrect numerical values. Since the original statement did not include any numerical values, whether you used any or not, we did not grade on value correctness but on the correct approach. We were very lenient on this problem as long as the approach is correct.

Part (c)

For this part we just need to remember the equation relating the shear stress to the radius:

$$\tau = \frac{T\rho}{I_p} \quad \text{where} \quad I_p = \frac{\pi}{2}r^4 \tag{27}$$

Thus, for member AB

$$\tau_{AB} = \frac{T_{AB}r_{AB}}{\frac{\pi}{2}r_{AB}^4} = \tau_{\text{max}} = 105 \times 10^6 \text{ Pa} \tag{28}$$

$$r_{AB} = 0.0084 \text{ m} \tag{29}$$

$$\boxed{d_{AB} = 0.001693 \text{ m}} \tag{30}$$

For member DC

$$\tau_{DC} = \frac{T_{DC} r_{DC}}{\frac{\pi}{2} r_{DC}^4} = \tau_{\text{max}} = 105 \times 10^6 \text{ Pa} \tag{31}$$

$$r_{DC} = 0.00546 \text{ m} \tag{32}$$

$$\boxed{d_{DC} = 0.011737 \text{ m}} \tag{33}$$

Note, we did not take many points off for incorrect numerical values. There are various final results depending on what value of angular velocity you used. As mentioned, you could have used any angular velocity ranging from 350-400 for full credit. We have focused more on the approach for this problem.

Problem 3:

α_{Cu}	16 x 10^{-6}/K
E_{Cu}	120 GPa
ν_{Cu}	0.35
σ_y	200 MPa
α_{Si}	3 x 10^{-6}/K
Room temperature	20°C
T_o	100°C

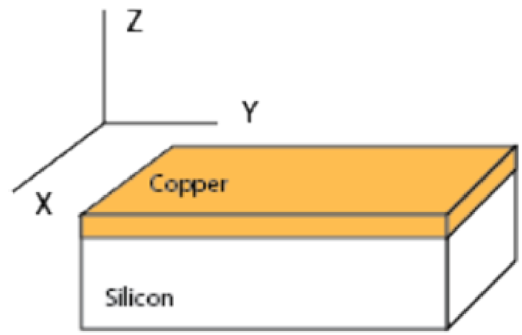

Figure 3

At the heart of your iPhone or laptop, there is a small microprocessor. This microprocessor consists essentially of a rectangular piece of silicon coated with many layers of other materials such as copper (to carry the signals) and silicon dioxide (serves as dielectric). Residual stresses in these coatings are a major reliability concern of manufacturers such as Intel or AMD. Let's try to estimate the residual stresses that develop in a thin layer of copper on a silicon substrate.

a. Assume that we have a thick silicon substrate and a very thin layer of copper as indicated in the figure. Both silicon and copper are stress-free at room temperature. When you turn on your iPhone or laptop, the temperature of the microprocessor increases from room temperature to T_o. If the thermal expansion coefficient of silicon is α_{Si} and that of Cu is α_{Cu}, where $\alpha_{Cu} >> \alpha_{Si}$ do you develop compressive or tensile normal stresses in the copper? Why?

b. Estimate the stress in the copper coating at T_o using the data in the table below. Assume that the copper is isotropic and elastic, and that the same thermal strain develops in all directions in the plane of the coating. Further assume that the stress perpendicular to the coating is zero and that the elastic deformation of the substrate is negligible.

c. What happens if the temperature, T_o , is so large that the stress in the copper in absolute value exceeds the yield stress of the copper? In that case, what is the stress in the copper at T_o ? What happens when you turn off your iPhone or laptop and the chip cools down to room temperature? Make a sketch of stress in the copper as a function of temperature.

Solution 3

Before we start, let's define our notations clearly here: ϵ_{xx}^{Cu} indicates the total strain of the copper layer, $\left(\epsilon_{xx}^{Cu}\right)_T$ is the stress of copper layer induced by thermal expansion and $\left(\epsilon_{xx}^{Cu}\right)_E$ is the strain of copper layer by elastic deformation. The similar definition applies for the silicon layer.

Part (a)

$$\begin{aligned}\left(\epsilon_{xx}^{Cu}\right)_T &= \alpha_{Cu}\Delta T = \alpha_{Cu}(T_0 - T_R)\\ \left(\epsilon_{xx}^{Si}\right)_T &= \alpha_{Si}\Delta T = \alpha_{Si}(T_0 - T_R)\end{aligned} \tag{34}$$

since $\alpha_{Cu} >> \alpha_{Si}$, as we increase the temperature, we would image that the copper layer tends to expand much more than the silicon substrate. While because the copper layer is coated on the silicon substrate which expands much less, the silicon substrate is actually "dragging" the copper layer back. Thus we conclude that there is **compressive normal stress** within copper layer.

Note: stating compressive stress with proper explanation gets 5pts.

Part (b)

We know that the total strain is the summation of thermal strain and elastic strain, thus,

$$\begin{aligned}\epsilon_{xx}^{Cu} &= \left(\epsilon_{xx}^{Cu}\right)_T + \left(\epsilon_{xx}^{Cu}\right)_E \\ \epsilon_{xx}^{Si} &= \left(\epsilon_{xx}^{Si}\right)_T + \left(\epsilon_{xx}^{Si}\right)_E\end{aligned} \tag{35}$$

According to the problem description *the elastic deformation of the substrate is negligible*, we know that

$$\left(\epsilon_{xx}^{Si}\right)_E = 0 \tag{36}$$

therefore,

$$\epsilon_{xx}^{Si} = \left(\epsilon_{xx}^{Si}\right)_T \tag{37}$$

Furthermore, as the copper layer and the silicon substrate are safely bonded, their total strains should be the same, i.e.,

$$\epsilon_{xx}^{Si} = \epsilon_{xx}^{Cu} \tag{38}$$

Combining Eq. 35, Eq. 37, and Eq. 38, we know that

$$\left(\epsilon_{xx}^{Cu}\right)_E = \left(\epsilon_{xx}^{Cu}\right)_T - \left(\epsilon_{xx}^{Si}\right)_T \tag{39}$$

From the description *the stress perpendicular to the coating is zero* we know that we are looking at a plane stress problem. Since both the copper layer and the silicon substrate are isotropic, we now generalize our result to y direction and considering Eq.(34),

$$\begin{aligned}\left(\epsilon_{xx}^{Cu}\right)_E &= \left(\epsilon_{xx}^{Cu}\right)_T - \left(\epsilon_{xx}^{Si}\right)_T = \left(\alpha_{Cu} - \alpha_{Si}\right)\left(T_0 - T_R\right) \\ \left(\epsilon_{yy}^{Cu}\right)_E &= \left(\epsilon_{yy}^{Cu}\right)_T - \left(\epsilon_{yy}^{Si}\right)_T = \left(\alpha_{Cu} - \alpha_{Si}\right)\left(T_0 - T_R\right)\end{aligned} \tag{40}$$

Now, for plane stress problem, the Generalized Hooke's Law can be deducted to,

$$\begin{aligned}\left(\epsilon_{xx}^{Cu}\right)_E &= \frac{1}{E}\left(\sigma_{xx}^{Cu} - \nu_{Cu}\,\sigma_{yy}^{Cu}\right) \\ \left(\epsilon_{yy}^{Cu}\right)_E &= \frac{1}{E}\left(\sigma_{yy}^{Cu} - \nu_{Cu}\,\sigma_{xx}^{Cu}\right)\end{aligned} \tag{41}$$

Our goal is to solve for the stress in the copper layer, i.e., solve for σ_{xx}^{Cu} and σ_{yy}^{Cu}, substitute eq. (40) into the plane stress equations yields the equation for stresses:

$$\begin{aligned}\left(\alpha_{Cu} - \alpha_{Si}\right)\left(T_0 - T_R\right) &= \frac{1}{E}\left(\sigma_{xx}^{Cu} - \nu_{Cu}\,\sigma_{yy}^{Cu}\right) \\ \left(\alpha_{Cu} - \alpha_{Si}\right)\left(T_0 - T_R\right) &= \frac{1}{E}\left(\sigma_{yy}^{Cu} - \nu_{Cu}\,\sigma_{xx}^{Cu}\right)\end{aligned} \tag{42}$$

solve for stresses,

$$\boxed{\sigma_{xx}^{Cu} = \sigma_{yy}^{Cu} = \frac{E\left(\alpha_{Cu} - \alpha_{Si}\right)\left(T_0 - T_R\right)}{1 - \nu_{Cu}} = 192 \text{ MPa}} \tag{43}$$

the stress calculated is less than yield stress.

Note: writing proper thermal expansion equations, strain-stress relations gets 2pts; solving the 1D (should be 2D) problem correctly gets 3pts; mentioning 2D plane stress problem with proper derivation about thermal expansion gets 4pts; obtaining the final stress expression gets 5pts.

Part (c)

If T_0 is so large that the stress in the copper in absolute value exceeds the yield stress of the copper, the copper begins to [plastically deform] (1pt). The stress in copper will be [equal to the yield strength $-\sigma_Y$] (1pt) beyond that temperature.
As your turn off your iPhone, the chips cool down to room temperature. Since the silicon substrate will go back to original shape, the copper layer will also go back to zero strain, with certain [residue stress] (1pts).

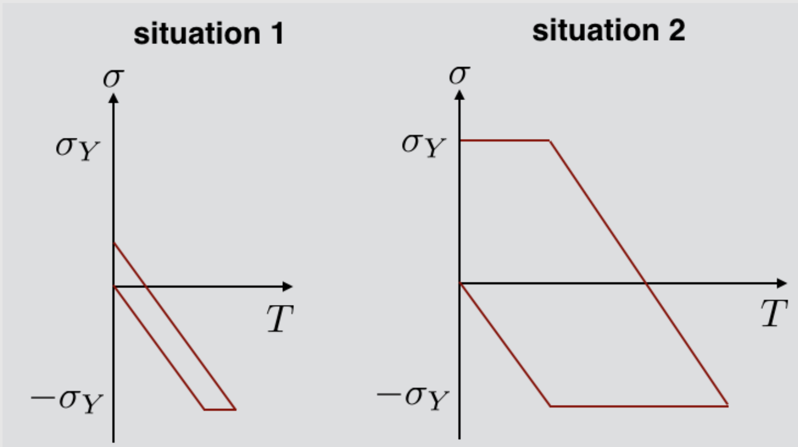

Figure 4

Note: plot either 1 or 2 is considered as full score.