

Engineering Electromagnetics Essentials

Chapter 5

*Basic concepts of time-varying
electric and magnetic fields*

Objective

Development of concepts in time varying electric and magnetic fields

Topics dealt with

Continuity equation and relaxation time

Faraday's law in integral and differential forms

Electromagnetic induction

Appreciation of coupling between electric and magnetic fields

Displacement current

Maxwell's equations in integral and differential forms

Relation between electric and magnetic fields in time-varying situations

Lorentz condition and wave equations in electric scalar and magnetic vector potentials as well as their solutions keeping in view their potential applications

Background

Vector calculus expressions developed in Chapter 2

Basic concepts of static electric and magnetic fields developed in Chapters 3 and 4 respectively

Continuity equation

Consider a flow of electrical charge through a region that constitutes a dc electric current.

Surround a point in the region of charge flow with a closed volume.

Flux of current density vector through the surface area S constitutes the current flowing out through the surface enclosure :

$$\text{Current} = \text{Flux of current density vector} = \oint_S \vec{J} \cdot d\vec{S} = \oint_S \vec{J} \cdot \vec{a}_n dS$$

(\vec{J} = current density vector)

For dc current (not varying with time), the charge flowing into the volume enclosure must be equal to the charge flowing out giving

$$\oint_S \vec{J} \cdot \vec{a}_n dS = 0$$

What is the corresponding finding for ac current (varying with time)?

Consider a flow of electrical charge through a region that constitutes an ac electric current.

For ac current (varying with time), the charge flowing into the volume enclosure is **not equal to** the charge flowing out.

The amount of charge in the volume enclosure must decrease with time if the amount of outflow of charge exceeds that of inflow, while the amount of charge in the volume enclosure must increase with time if the amount of charge inflow exceeds that of charge outflow.

For a time-varying (ac) current, equate the flux of current density vector through the volume enclosure, interpreted as the flux going out of the volume, to the negative time rate of change of charge Q in the volume:

$$\text{Flux of } \vec{J} = \oint_S \vec{J} \cdot \vec{a}_n dS = -\frac{dQ}{dt}$$

$$\text{Flux of } \vec{J} = \oint_S \vec{J} \cdot \vec{a}_n dS = -\frac{dQ}{dt} \quad \longleftarrow \quad Q = \int_{\tau} \rho d\tau \quad (\rho \text{ is the volume charge density})$$

For a positive value of the flux of current density (left hand side), dQ/dt (time rate of change of charge within the enclosure) in the right hand side has to be negative corresponding to a decrease in charge within the enclosure.

$$\oint_S \vec{J} \cdot \vec{a}_n dS = -\frac{d}{dt} \int_{\tau} \rho d\tau$$

Choose the surface area of the volume enclosing the charge to be constant. Since the integral on the right hand side is convergent, we may replace the complete derivative in the right hand side by its partial derivative and put it under the definite integral

$$\int_S \vec{J} \cdot \vec{a}_n ds = \int_{\tau} -\frac{\partial \rho}{\partial t} d\tau$$

$$\int_S \vec{J} \cdot \vec{a}_n ds = \int_{\tau} -\frac{\partial \rho}{\partial t} d\tau$$

$$\int_S \vec{J} \cdot \vec{a}_n ds \cong -\frac{\partial \rho}{\partial t} \int_{\Delta \tau} d\tau \quad \longleftarrow$$



$$\int_S \vec{J} \cdot \vec{a}_n ds \cong -\frac{\partial \rho}{\partial t} \Delta \tau$$



$$\frac{\int_S \vec{J} \cdot \vec{a}_n ds}{\Delta S} \cong -\frac{\partial \rho}{\partial t}$$

If we take the volume element to be an infinitesimal volume $\Delta \tau$, we may regard the volume charge density ρ to be approximately constant within the infinitesimal volume and take its time derivative outside the integral. This approximation leads to putting the sign of equality as the sign of approximate equality.

$$\frac{\int_{\Delta S} \vec{J} \cdot \vec{a}_n ds}{\Delta \tau} \cong -\frac{\partial \rho}{\partial t}$$



The relation becomes exact if the volume element $\Delta \tau$ shrinks to zero, making it more reasonable to assume the volume charge density to be constant within the volume enclosure .

$$\lim_{\Delta \tau \rightarrow 0} \frac{\int_S \vec{J} \cdot \vec{a}_n ds}{\Delta \tau} = -\frac{\partial \rho}{\partial t}$$

By definition the left hand side is the divergence of current density: $\nabla \cdot \vec{J}$



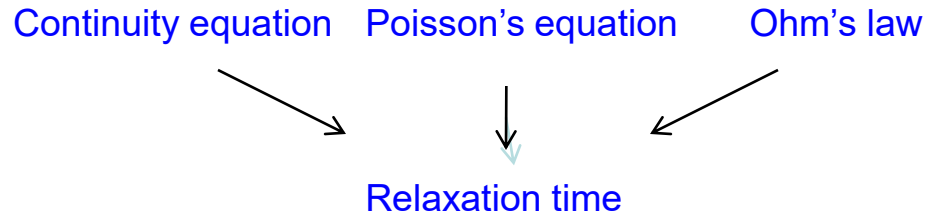
$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$



$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad (\text{continuity equation})$$

Relaxation time

Relaxation time is a measure of how fast or slow a medium of uniform conductivity and permittivity approaches electrostatic equilibrium.



$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \longleftarrow \quad \vec{J} = \sigma \vec{E}$$

↓ Conductivity is uniform in the medium

$$\sigma(\nabla \cdot \vec{E}) + \frac{\partial \rho}{\partial t} = 0 \quad \longleftarrow \quad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon}$$

$$\sigma\left(\frac{\rho}{\epsilon}\right) + \frac{\partial \rho}{\partial t} = 0$$

$$\frac{\partial \rho}{\partial t} = -\frac{\sigma}{\epsilon} \rho$$

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$$\int \frac{d\rho}{\rho} = -\frac{\sigma}{\varepsilon} \int dt$$



$$\ln \rho = -\frac{\sigma}{\varepsilon} t + \text{constant}$$



$$\ln \rho = -\frac{\sigma}{\varepsilon} t + \ln \rho_0$$

$$\ln \rho - \ln \rho_0 = -\frac{\sigma}{\varepsilon} t$$



$$\ln \frac{\rho}{\rho_0} = -\frac{\sigma}{\varepsilon} t$$

We tacitly choose constant
in terms of another constant ρ_0

$$\text{constant} = \ln \rho_0$$

$$\ln \frac{\rho}{\rho_0} = -\frac{\sigma}{\varepsilon} t$$

↓

$$\frac{\rho}{\rho_0} = \exp\left(-\frac{\sigma}{\varepsilon} t\right)$$

↓

$$\rho = \rho_0 \exp\left(-\frac{\sigma}{\varepsilon} t\right)$$

↓

$$\rho = \rho_0 \exp\left(-\frac{t}{T}\right) \longleftarrow \text{Relaxation time } T = \frac{\varepsilon}{\sigma} = \frac{\varepsilon_r \varepsilon_0}{\sigma}$$

For very large values of
relaxation time T

$$\left. \begin{array}{l} -\frac{t}{T} \rightarrow 0 \\ \exp\left(-\frac{t}{T}\right) \rightarrow 1 \\ \rho \rightarrow \rho_0 \end{array} \right\}$$

For very small values of
relaxation time T

$$\left. \begin{array}{l} -\frac{t}{T} \rightarrow -\infty \\ \exp\left(-\frac{t}{T}\right) \rightarrow 0 \\ \rho \rightarrow 0 \end{array} \right\}$$

For very large values of relaxation time T

$$\left. \begin{aligned} T &= \frac{\varepsilon}{\sigma} = \frac{\varepsilon_r \varepsilon_0}{\sigma} \\ -\frac{t}{T} &\rightarrow 0 \\ \exp\left(-\frac{t}{T}\right) &\rightarrow 1 \\ \rho &\rightarrow \rho_0 \end{aligned} \right\}$$

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For a dielectric medium, the value of conductivity σ is very small that renders T a very large value. This makes the volume charge density ρ in the bulk of the dielectric tend to ρ_0 (equilibrium volume charge density). Therefore, within a time of interest t , the bulk of a dielectric medium can be charged with the equilibrium volume charge density (ρ_0).

On the other hand, for a medium of good conductivity, the value of conductivity σ is very large that renders T a very small value. This makes the volume charge density ρ in the bulk of the dielectric tend to 0. Thus, the bulk of a medium of a good conductor cannot be charged; any charge injected into such a medium of good conductivity will not stay long within the bulk of the conductor only to reappear at the outer surface of the conducting medium in compliance with the requirement of the conservation of charge.

**Conductivity, permittivity, and
relaxation time of typical
medium materials**

Medium material	σ (mho/m)	ϵ_r	$T = \epsilon_r \epsilon_0 / \sigma$
Copper	5.8×10^7	1	1.5×10^{-19} s
Sea-water	4	81	2×10^{-10} s
Corn oil	5×10^{-4}	3.1	0.55 s
Mica	10^{-15}	5.8	~1/2 a day
Quartz (fused)	10^{-17}	5	~50 days

The concept of the relaxation time is very useful in understanding the electromagnetic boundary conditions at the interface between two dielectrics as well as those at the interface between a conductor and a dielectric.

Time-varying magnetic field and Faraday's law of electromagnetic induction

Electromagnetic force (EMF)

Electromagnetic force (EMF) is a measure of the capability of a source of energy to drive an electric charge around a circuit, that is, generate a circuit current; it is estimated as the energy per unit charge that is imparted by the energy source. The unit of EMF is, therefore, that of potential, that is, volt or V.

Faraday's law

Faraday's law relates the time rate of change of magnetic flux linked with a closed circuit with an electromotive force (EMF) induced in the circuit.

$$i = \frac{\mathcal{E} + \mathcal{E}_{\text{induced}}}{R}$$

(Faraday's law)

$$\left\{ \begin{array}{l} \mathcal{E} = \text{EMF of the source} \\ \mathcal{E}_{\text{induced}} = \text{EMF induced in the circuit} \end{array} \right.$$

$R =$ Resistance of the circuit

$$i = \frac{\mathcal{E} + \mathcal{E}_{\text{induced}}}{R}$$

(Faraday's law)

$\left\{ \begin{array}{l} \mathcal{E} = \text{EMF of the source} \\ \mathcal{E}_{\text{induced}} = \text{EMF induced in the circuit} \end{array} \right.$
 $R = \text{Resistance of the circuit}$

We can also express Faraday's law in the following form

$$i = \frac{\mathcal{E} + \mathcal{E}_{\text{induced}}}{R}$$

$\phi_B = \text{Flux of magnetic flux density linked with the circuit}$

$$\mathcal{E}_{\text{induced}} = -\frac{d\phi_B}{dt}$$

$$i = \frac{\mathcal{E} - \frac{d\phi_B}{dt}}{R} \quad (\text{Faraday's law})$$

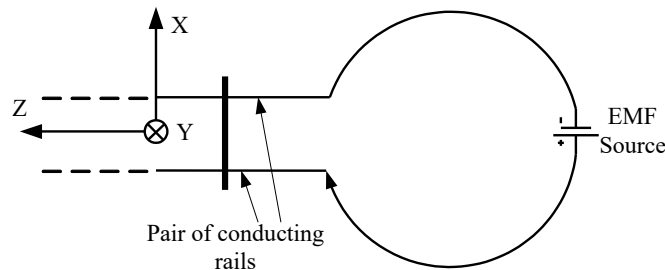
How can we appreciate

$$\mathcal{E}_{\text{induced}} = -\frac{d\phi_B}{dt} ?$$

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In order to appreciate that the induced voltage is the negative time derivative of magnetic flux linked with a circuit, as it is given by the above expression, let us consider a conducting rod which is free to slide on a pair of transverse conducting rails, the whole immersed in a uniform steady normal magnetic field while a current passes through the circuit from a source of electromagnetic force.



The force due to the applied magnetic field on the conducting rod carrying current supplied by a source of EMF causes a motion of the conducting rod on the pair of conducting rails.

Let the rod, rail and circuit lie on ZX plane and the magnetic field be applied along y direction.

The energy supplied by the source of EMF is balanced by the energy that is spent in doing work to move the conducting rod plus the energy that is lost in the circuit resistance.

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$$\Delta W = (\Delta W)_1 + (\Delta W)_2$$

$$\Delta W = \mathcal{E} \Delta Q = \mathcal{E} i \Delta t$$

$$\Delta Q = i \Delta t$$

$$(\Delta W)_1 = i^2 R \Delta t$$

$$(\Delta W)_2 = \vec{F} \cdot \Delta z \vec{a}_z$$

$$\mathcal{E} i \Delta t = i^2 R \Delta t + \vec{F} \cdot \Delta z \vec{a}_z$$

ΔW = energy spent by the source of EMF for supplying circuit current i in time Δt or in other words charge $\Delta Q = i \Delta t$

$(\Delta W)_1$ = energy lost due to the circuit resistance R in time Δt

$(\Delta W)_2$ = energy spent in doing work to move the rod by the force \vec{F} on the rod due to the magnetic field through an element of distance Δz in time Δt

$$\vec{F} = i \int d\vec{l} \times \vec{B} = i \vec{l} \times \vec{B}$$

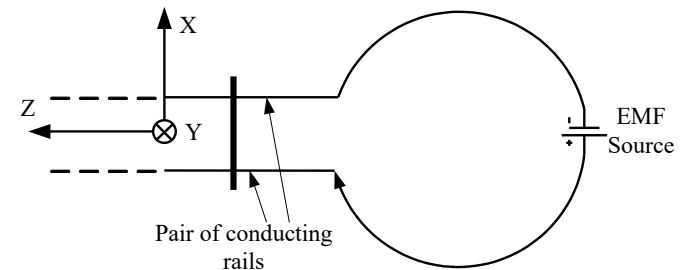
$$\vec{B} = B \vec{a}_y$$

$\vec{l} = l \vec{a}_x$ (current flow being along x in the conducting rod)

l = length of the rod intercepted by the circuit

$$\vec{F} = i \int d\vec{l} \times \vec{B} = i \vec{l} \times \vec{B} = i(l \vec{a}_x) \times (B \vec{a}_y) = ilB(\vec{a}_x \times \vec{a}_y) = ilB \vec{a}_z$$

$$\mathcal{E} i \Delta t = i^2 R \Delta t + \vec{F} \cdot \Delta z \vec{a}_z = i^2 R \Delta t + (ilB \vec{a}_z) \cdot (\Delta z \vec{a}_z) = i^2 R \Delta t + ilB \Delta z$$



$$\mathcal{E}i \Delta t = i^2 R \Delta t + \vec{F} \cdot \Delta z \vec{a}_z = i^2 R \Delta t + (ilB\vec{a}_z) \cdot (\Delta z \vec{a}_z) = i^2 R \Delta t + ilB\Delta z$$

$$\mathcal{E}i \Delta t = i^2 R \Delta t + ilB\Delta z$$

Divide by $i \Delta t$

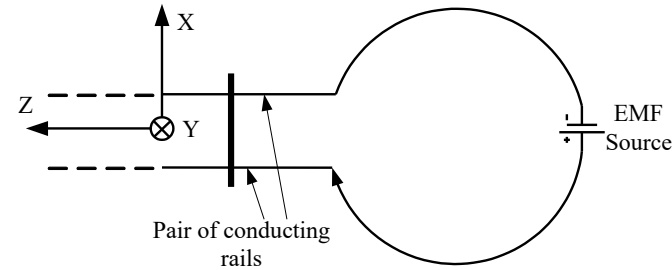
$$\mathcal{E} = iR + lB \frac{\Delta z}{\Delta t}$$

$$i = \frac{\mathcal{E} - lB \frac{\Delta z}{\Delta t}}{R}$$

$$i = \frac{\mathcal{E} - \frac{\Delta \phi_B}{\Delta t}}{R}$$

which becomes in the limit for instantaneous current $\Delta t \rightarrow 0$

$$i = \frac{\mathcal{E} - \frac{d\phi_B}{dt}}{R} \quad (\text{Faraday's law})$$



The rod moves an infinitesimal distance Δz in infinitesimal time Δt due to the force on it due to magnetic field along z : $\vec{F} = ilB\vec{a}_z$

This cause an increase in the area of the circuit by $\Delta S = l \Delta z$
Consequently, this causes an increase in flux of magnetic flux density

$$\Delta \phi_B = \vec{B} \cdot \vec{a}_n \Delta S$$

$$\Delta \phi_B = \vec{B} \cdot \vec{a}_n \Delta S = (B \vec{a}_y) \cdot (\vec{a}_y) \Delta S = B \Delta S$$

$$\Delta S = l \Delta z$$

$$\Delta \phi_B = B \Delta S = Bl \Delta z$$

$$\vec{B} = B \vec{a}_y$$

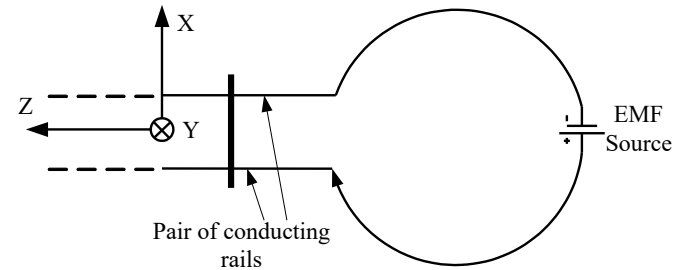
$$\vec{a}_n = \vec{a}_y$$

$$\frac{\Delta \phi_B}{\Delta t} = lB \frac{\Delta z}{\Delta t}$$

$$i = \frac{\mathcal{E} - \frac{d\phi_B}{dt}}{R} \quad (\text{Faraday's law})$$

$$\mathcal{E}_{\text{induced}} = -\frac{d\phi_B}{dt} \quad (\text{Faraday's law})$$

$$i = \frac{\mathcal{E} + \mathcal{E}_{\text{induced}}}{R} \quad (\text{Faraday's law})$$



While formulating the problem, we have assumed the current flow to be in positive x direction, causing a movement of the rod in positive z direction resulting in an increase in the area of the circuit, and consequently, an increase in the magnetic flux linked with the circuit that makes $d\phi_B/dt$ positive. You can then appreciate that if the direction of either the current or the magnetic field are reversed, then the force on the rod and its movement will also reverse and the circuit area as well as the magnetic flux linked with the circuit will decrease, thereby making $d\phi_B/dt$ negative. Therefore, in general, we can write

$$\left. \begin{aligned} i &= \frac{\mathcal{E} + \mathcal{E}_{\text{induced}}}{R} \\ i &= \frac{\mathcal{E} - \left| \frac{d\phi_B}{dt} \right|}{R} \\ \mathcal{E}_{\text{induced}} &= - \left| \frac{d\phi_B}{dt} \right| \end{aligned} \right\} \quad (\text{Faraday's law})$$

Lenz's law

$$\left. \begin{aligned} i &= \frac{\mathcal{E} + \mathcal{E}_{\text{induced}}}{R} \\ i &= \frac{\mathcal{E} - \left| \frac{d\phi_B}{dt} \right|}{R} \\ \mathcal{E}_{\text{induced}} &= - \left| \frac{d\phi_B}{dt} \right| \end{aligned} \right\} \text{(Faraday's law)}$$

You can recapitulate, referring to the example of the current carrying conducting rod on rail in a magnetic field dealt with, and with the help of the above expression for Faraday's law, that the circuit current decreases due to electromagnetic induction. Consequently, this would reduce the force on the conducting rod and oppose its movement that causes a change in the circuit area. This in turn would reduce the change in magnetic flux linked with the circuit responsible for electromagnetic induction. The phenomenon demonstrated in this finding is known as Lenz's law, according to which electromagnetic induction takes place such that it opposes the cause to which it is due; in this example, the cause is the movement of the rod which is opposed by the effect of the decrease of current in the rod due to electromagnetic induction.

Simple examples of the application of Faraday's law

In an example to illustrate Faraday's law consider a straight conducting rod of a given length that moves parallel to its length with a given velocity on a plane perpendicular to a given magnetic flux density of a uniform magnetic field region, and hence obtain an expression for the amount of voltage developed across the length of the rod, with the help of Faradays law, as follows.

$$\left. \begin{array}{l} l = \text{length of the rod} \\ v = \text{velocity of the rod} \\ B = \text{magnetic flux density (magnitude)} \end{array} \right\} \text{(given)}$$

$l\Delta z =$ infinitesimal area traced by the rod in infinitesimal time Δt

$\Delta z =$ infinitesimal distance traversed by the rod in infinitesimal time Δt

Element of magnetic flux $\Delta\phi_B$ through the element of area $l\Delta z$ is obtained by multiplying the element of area by the magnitude of magnetic flux density B as

$$\Delta\phi_B = Bl\Delta z$$

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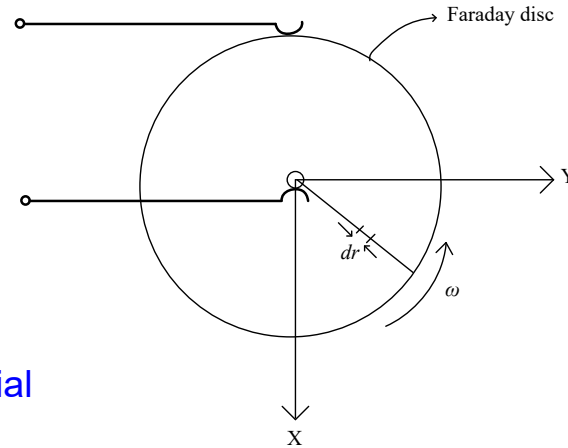


$$\frac{\Delta\phi_B}{\Delta t} = Bl \frac{\Delta z}{\Delta t}$$

$$\begin{aligned} \text{Amount of induced voltage} &= \frac{d\phi_B}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\phi_B}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{Bl\Delta z}{\Delta t} \\ &= Bl \left(\lim_{\Delta t \rightarrow 0} \frac{\Delta z}{\Delta t} \right) = Blv \end{aligned}$$

(expression for the amount of voltage developed across the length of the rod)

In another illustration to exemplify Faraday's law, let us find an expression for the amount of voltage developed at the rim with respect to that at the axis or axle of a conducting disc, called Faraday disc, rotating in a region of magnetic field is perpendicular to the disc. We can find the desired expression with the help of the preceding example as follows.



Element of voltage developed along an element of radial rod length dr considered at a radial distance r from the axis of the disc, will be $Bdrv$, where v is the velocity of the element of radial rod length (so found with the help of the result of the preceding example). In view of the relation $v = \omega r$, where ω is the angular velocity of rotation of the disc, we can then write

Element of voltage developed across dr (element of rod length) = $Bdr\omega r = B\omega dr^2$



Integrating between the axle at $r = 0$ and the rim at $r = a$ of the disc of radius a

Amount of induced voltage between the rim and axle

$$= \int_0^a B\omega dr^2 = B\omega \int_0^a r^2 dr = B\omega \left[\frac{r^3}{3} \right]_0^a = \frac{B\omega a^3}{3}$$

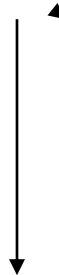
(expression for the voltage developed at the rim of the disc with respect to that at its axle)

Integral form of Faraday's law

The induced EMF in a circuit sets up an induced current, which can be attributed to a flow of charge in an induced electric field \vec{E}

If we move a test point charge q round the closed path coinciding with the circuit, the element of work done dW on the charge in moving it through an element of length dl of the path may be written as

$$dW = \vec{F} \cdot d\vec{l}$$



\vec{F} = force on a test charge q due to induced electric field \vec{E}

$$\vec{F} = q\vec{E}$$

$d\vec{l}$ = element of length vector on the circuit of magnitude dl whose direction is the direction in which the test charge q is moved

$$dW = q\vec{E} \cdot d\vec{l}$$



Integrating round the closed circuit

$$W = \oint_l dW = \oint_l q\vec{E} \cdot d\vec{l} = q \oint_l \vec{E} \cdot d\vec{l} \quad (\text{work done in moving the charge } q \text{ round the circuit})$$

$$W = \oint_l dW = \oint_l q\vec{E} \cdot d\vec{l} = q \oint_l \vec{E} \cdot d\vec{l} \quad (\text{work done in moving the charge } q \text{ round the circuit})$$

comparing W may also be equated to the energy imparted by the induced EMF to the charge q as

$$W = q \mathcal{E}_{\text{induced}}$$

$$q \oint_l \vec{E} \cdot d\vec{l} = q \mathcal{E}_{\text{induced}}$$

$$\mathcal{E}_{\text{induced}} = \oint_l \vec{E} \cdot d\vec{l}$$

comparing

$$\mathcal{E}_{\text{induced}} = -\frac{d\phi_B}{dt} \quad (\text{Faraday's law}) \quad (\text{obtained earlier})$$

$$\oint_l \vec{E} \cdot d\vec{l} = -\frac{d\phi_B}{dt} \quad (\text{Faraday's law})$$

$$\oint_l \vec{E} \cdot d\vec{l} = -\frac{d\phi_B}{dt} \quad \longleftarrow \quad \phi_B = \int_S \vec{B} \cdot \vec{a}_n dS$$

$$\oint_l \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot \vec{a}_n dS$$

In situations where the area remains constant and the magnetic flux density varies with time, we can replace the total time-derivative on the right-hand side by the partial derivative and put it under a convergent, definite integral.

$$\oint_l \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{a}_n dS$$

(Faraday's law)

Faraday's law in integral form expressed in different ways may be put together as follows:

$$\mathcal{E}_{\text{induced}} = -\frac{d\phi_B}{dt} \text{ (Faraday's law)}$$



$$\left. \begin{aligned} \oint_l \vec{E} \cdot d\vec{l} &= -\frac{d\phi_B}{dt} \\ \oint_l \vec{E} \cdot d\vec{l} &= -\frac{d}{dt} \int_S \vec{B} \cdot \vec{a}_n dS \\ \oint_l \vec{E} \cdot d\vec{l} &= -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{a}_n dS \end{aligned} \right\} \text{ (Faraday's law in integral form)}$$

That electric and magnetic fields are coupled in time-varying phenomena is evident from Faraday's law.

Differential form of Faraday's law

We have earlier derived from first principles (in Chapter 4) the differential form of Ampere's circuital law from its integral form. On the same line, we can also obtain the differential form of Faraday's law from its integral form. In fact, we need not now repeat the derivation once we identify the analogous quantities in Ampere's circuital and Faraday's laws. Let us see how this can be done.

Let us recall (from Chapter 4) Ampere's circuital law as follows:

$$\oint_l \vec{H} \cdot d\vec{l} = i \quad (\text{Ampere's circuital law})$$

$$i = \int_S \vec{J} \cdot \vec{a}_n dS$$

$$\oint_l \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot \vec{a}_n dS \quad (\text{Ampere's circuital law})$$

comparing

Let us recall Faraday's law as follows:

$$\oint_l \vec{E} \cdot d\vec{l} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{a}_n dS = \int_S - \frac{\partial \vec{B}}{\partial t} \cdot \vec{a}_n dS \quad (\text{Faraday's law})$$

\vec{E} is analogous to \vec{H} and

\vec{J} and is analogous to $-\frac{\partial \vec{B}}{\partial t}$

$$\oint_l \vec{H} \cdot d\vec{l} = \int_s \vec{J} \cdot \vec{a}_n dS$$

(Ampere's circuital law)

$$\oint_l \vec{E} \cdot d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot \vec{a}_n dS = \int_s -\frac{\partial \vec{B}}{\partial t} \cdot \vec{a}_n dS$$

(Faraday's law)

Comparing the two laws we have already identified the analogous quantities:

\vec{E} is analogous to \vec{H} and \vec{J} and is analogous to $-\frac{\partial \vec{B}}{\partial t}$

We have already derived (in Chapter 4)

$$\nabla \times \vec{H} = \vec{J} \quad \text{from} \quad \oint_l \vec{H} \cdot d\vec{l} = \int_s \vec{J} \cdot \vec{a}_n dS$$

Therefore, without repeating the derivation we can write analogously

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{from} \quad \oint_l \vec{E} \cdot d\vec{l} = \int_s -\frac{\partial \vec{B}}{\partial t} \cdot \vec{a}_n dS$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (\text{Faraday's law in differential form})$$

That electric and magnetic fields are coupled in time-varying phenomena is evident from the association of electric field with time-varying magnetic field given by integral or differential form of Faraday's law.

Furthermore, magnetic field is also associated with time-varying electric field, which however is not given by Faraday's law and which can be appreciated from the concept of displacement current to follow.

Time-varying electric field and Displacement Current

Let us see how magnetic field is associated with a time-varying electric field.

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad (\text{continuity equation}) \quad \vec{D} = \text{electric displacement}$$

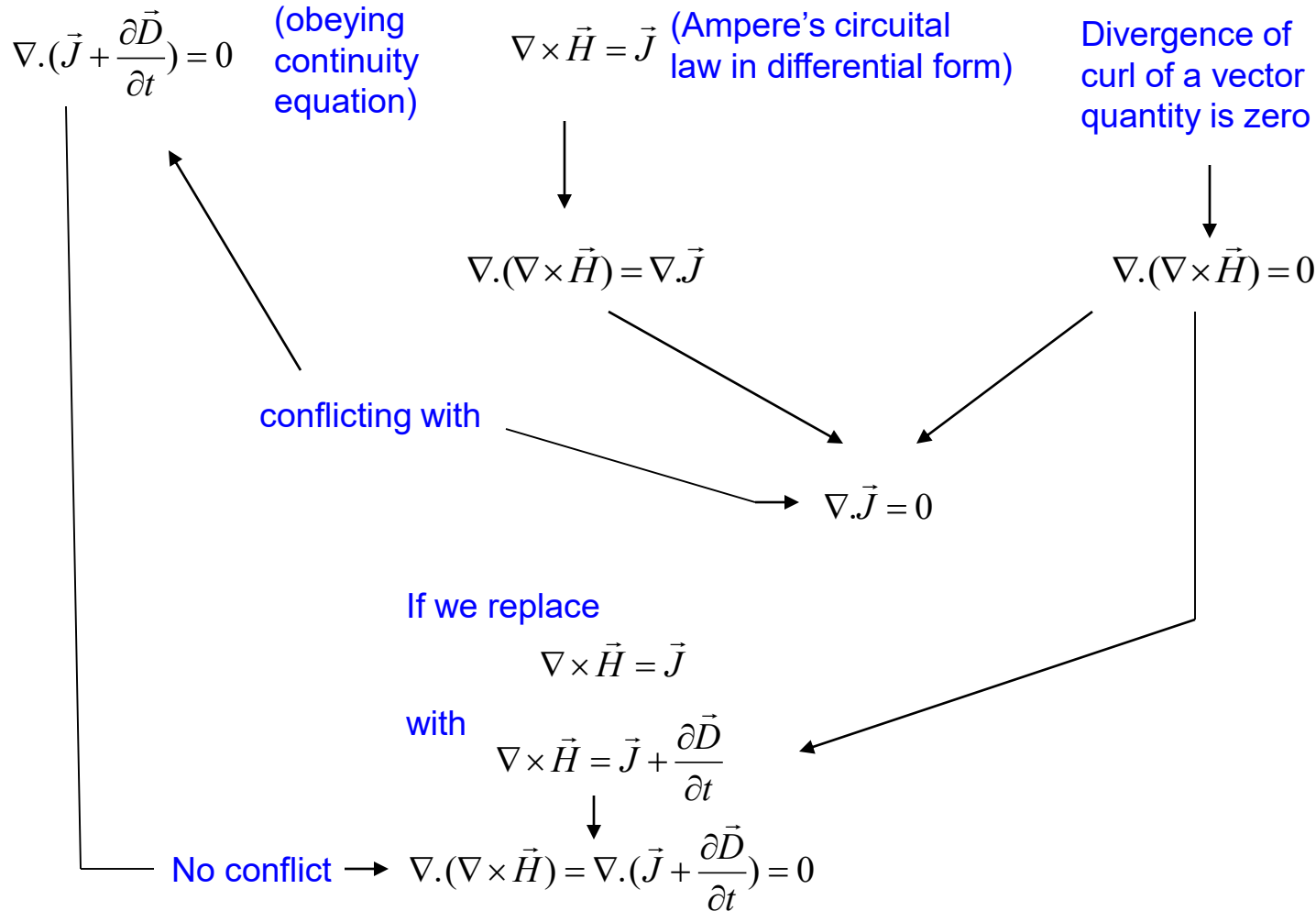
$$\nabla \cdot \vec{D} = \rho \quad (\text{Poisson's equation})$$

$$\nabla \cdot \vec{J} + \frac{\partial}{\partial t} \nabla \cdot \vec{D} = 0$$

The partial time-derivative and the divergence, which involves partial derivatives with respect to space/angle coordinates, are interchangeable.

$$\nabla \cdot \vec{J} + \nabla \cdot \frac{\partial \vec{D}}{\partial t} = 0$$

$$\nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0 \quad (\text{obeying continuity equation})$$



Thus, we get the differential form of Ampere's circuital law that is consistent with the continuity equation in time-varying situations as

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (\text{differential form of Ampere's circuital law that is consistent with the continuity equation in time-varying situations})$$

(showing how magnetic field is associated with time varying electric field) ($\vec{D} = \epsilon \vec{E}$)

$\vec{J} =$ Conduction current density or convection current density due to the flow of charge in a medium depending on whether the medium is a conductor or free space

$\frac{\partial \vec{D}}{\partial t} =$ Displacement current density not constituted by the flow of charge in a free-space or a dielectric medium if, with time, there is a variation of electric field hence electric displacement ($\vec{D} = \epsilon \vec{E}$)

Thus, the displacement current density in a medium is given by the partial derivative of electric displacement with time for a time-varying electric field.

A capacitor blocks dc current but allows ac current to pass through it. The ac current through a capacitor is the displacement current caused by time-varying electric field in the capacitor due to an ac voltage applied on it.

Let us appreciate with reference to the example of a parallel-plate capacitor that the current density in the capacitor is equal to the displacement current density.

$$\begin{aligned}
 i_{\text{capacitor}} &= \frac{dQ}{dt} = \text{current through capacitor} \\
 &\downarrow \\
 i_{\text{capacitor}} &= \frac{dQ}{dt} = C \frac{dV}{dt} && \begin{aligned} &Q = CV && \leftarrow C = \text{capacitance} \\ & && \leftarrow V = \text{potential difference between plates} \end{aligned} \\
 &\downarrow \\
 i_{\text{capacitor}} &= Cd \frac{dE}{dt} && \begin{aligned} &V = Ed && \leftarrow E = \frac{V}{d} = \text{electric field between plates} \end{aligned} \\
 &\downarrow \\
 i_{\text{capacitor}} &= \varepsilon A \frac{dE}{dt} && \begin{aligned} &C = \frac{\varepsilon A}{d} && \leftarrow \begin{aligned} &A = \text{plate area} \\ &d = \text{distance between plates} \end{aligned} \\ & && \leftarrow \varepsilon = \text{permittivity of the medium between plates} \end{aligned} \\
 &\downarrow \\
 i_{\text{capacitor}} &= A \frac{dD}{dt} && \leftarrow D = \varepsilon E \\
 &\downarrow \\
 J_{\text{capacitor}} &= \frac{i_{\text{capacitor}}}{A} = \frac{dD}{dt}
 \end{aligned}$$

$$J_{\text{capacitor}} = \frac{i_{\text{capacitor}}}{A} = \frac{dD}{dt}$$

(capacitor current density)

↑
comparing

Displacement current density is given by

$$J_{\text{displacement}} = \frac{\partial D}{\partial t}$$

Within the capacitor the electric field and electric displacement are each independent of space coordinates. Replacing the partial derivative with complete derivative we then have

$$J_{\text{displacement}} = \frac{dD}{dt}$$

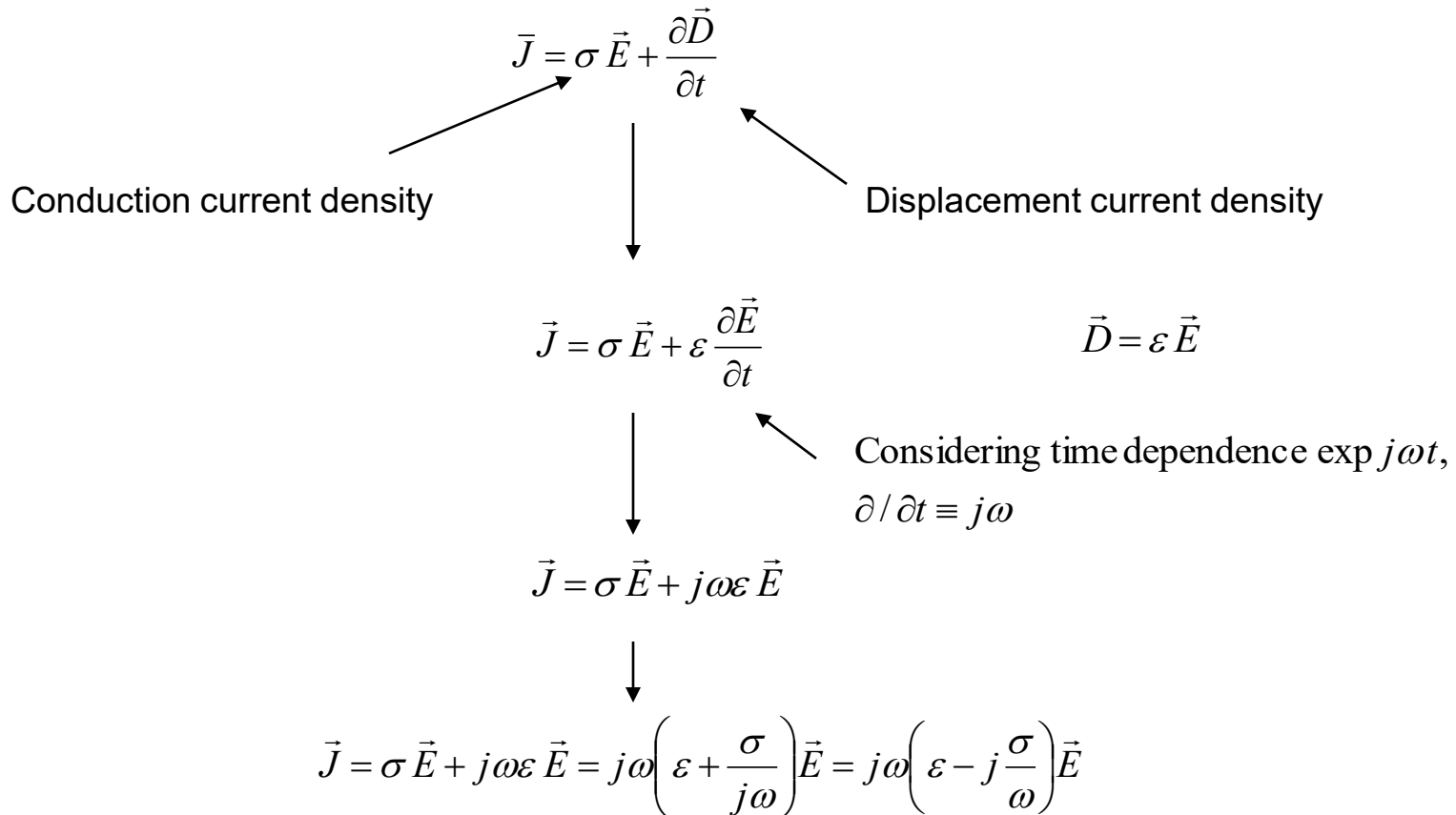
↓

$$J_{\text{capacitor}} = J_{\text{displacement}}$$

(Capacitor current density is thus identified as the displacement current density)

Complex permittivity and loss tangent

Current through an imperfect or lossy dielectric comprises conduction and displacement currents.



$$\vec{J} = j\omega \left(\epsilon - j \frac{\sigma}{\omega} \right) \vec{E}$$

$$\begin{aligned} \epsilon' &= \epsilon \\ \epsilon'' &= \frac{\sigma}{\omega} \end{aligned}$$

$$\vec{J} = j\omega(\epsilon' - j\epsilon'')\vec{E}$$

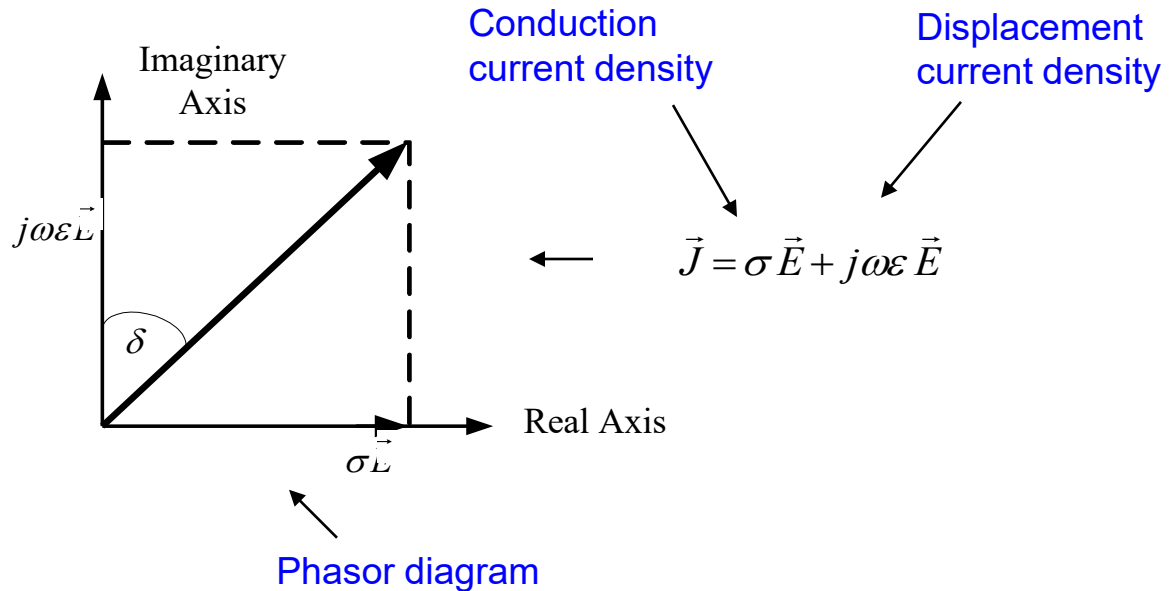
$$\epsilon^* = \epsilon' - j\epsilon'' \quad \leftarrow \text{Complex permittivity } \epsilon^*$$

$$\vec{J} = j\omega\epsilon^*\vec{E} \quad (\text{current density in terms of electric field and complex permittivity})$$

δ is the angle by which the current density in an imperfect dielectric ($\sigma \neq 0$) falls short of leading the electric field by the phase angle of $\pi/2$

$$\tan \delta = \frac{\epsilon''}{\epsilon'} = \frac{\sigma / \omega}{\epsilon} = \frac{\sigma}{\omega\epsilon}$$

(Loss tangent)



Maxwell's equations

Simple enough to imprint on a T-shirt and yet rich enough to provide new insights throughout a lifetime of study — J.R. Whinnery

“The teaching of electromagnetics”, *IEEE Trans. Education* **ED-33** (1990) p.327

Several disciplines hang as gems on one priceless necklace which it was Maxwell's privilege and honour to recognize as capricious Nature's enduring ornament

— P. Khastagir

“Apologia,” *Seminar on Electromagnetics and their applications*, 22-23 December 1988, Varanasi, India

James Clerk Maxwell originally gave as many as twenty equations in twenty variables; it was Oliver Heaviside, who is one of the founders of vector calculus, who reduced these equations to four equations.

We may recall the following four equations in their integral and differential forms; they have already been derived and are known as Maxwell's equations.

Maxwell's equations

Integral form	Differential form
$\oint_S \vec{D} \cdot d\vec{S} = \oint_S \vec{D} \cdot \vec{a}_n dS = \int_{\tau} \rho d\tau$	$\nabla \cdot \vec{D} = \rho$
$\oint_S \vec{B} \cdot d\vec{S} = \oint_S \vec{B} \cdot \vec{a}_n dS = 0$	$\nabla \cdot \vec{B} = 0$
$\oint_l \vec{E} \cdot d\vec{l} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{a}_n dS$	$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$
$\oint_l \vec{H} \cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{a}_n dS$	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

Maxwell's four equations are used extensively to develop concepts of electromagnetic theory and their applications.

Take up the following example to illustrate one of Maxwell's equations in its both integral and differential forms. The problem in this example is to find the magnetic field inside a parallel-plate capacitor consisting of circular plates in terms of the time rate of change of spatially uniform time-varying electric field inside the capacitor.

In one of the two approaches, one of Maxwell's equations is used in its integral form (approach I) while, in another approach, the same Maxwell's equation is used but now in its differential form (approach II).

In approach I, use the following Maxwell's equation in integral form:

$$\oint_l \vec{H} \cdot d\vec{l} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{a}_n dS$$

$\vec{J} = 0$

$$\oint_l \vec{H} \cdot d\vec{l} = \int_s \frac{\partial \vec{D}}{\partial t} \cdot \vec{a}_n dS$$

(perfect non-conducting dielectric between plates)

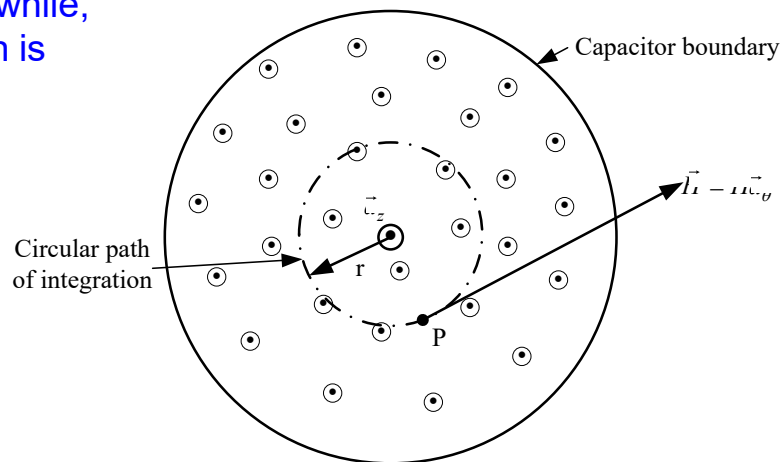
In approach II, use the following Maxwell's equation in differential form:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$\vec{J} = 0$

(perfect non-conducting dielectric between plates)

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$



Cross section perpendicular to Z axis of a parallel-plate capacitor consisting of circular plates showing the instantaneous time varying electric field represented by dots with circles (pointing towards the reader) and the corresponding azimuthal magnetic field at a point P on a coaxial circular path of radius r .

Approach I:

$$\vec{D}(=\epsilon \vec{E}) = D \vec{a}_z$$

$$\vec{a}_n = \vec{a}_z$$

$$\oint_l \vec{H} \cdot d\vec{l} = \int_s \frac{\partial \vec{D}}{\partial t} \cdot \vec{a}_n dS$$

$$\vec{H} = H_\theta \vec{a}_\theta$$

$$d\vec{l} = dl \vec{a}_\theta$$

$$\oint_l H_\theta \vec{a}_\theta \cdot dl \vec{a}_\theta = \int_s \frac{\partial D \vec{a}_z}{\partial t} \cdot \vec{a}_z dS$$

$$\left. \begin{aligned} \oint_l dl &= 2\pi r \\ \int_s dS &= \pi r^2 \end{aligned} \right\}$$

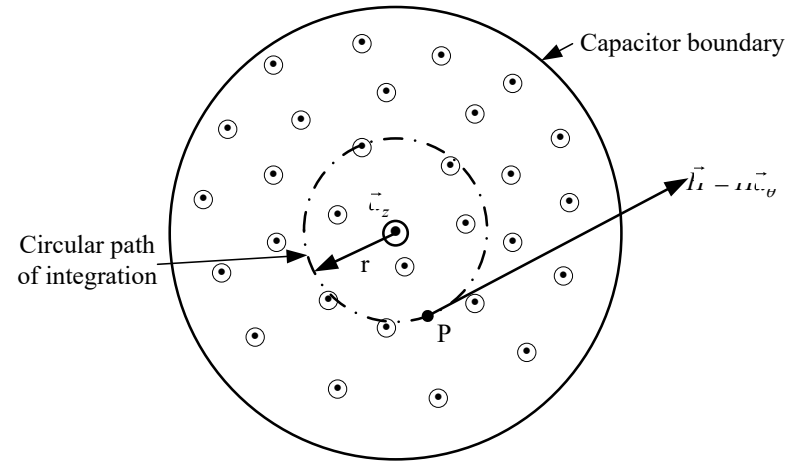
$$H_\theta \oint_l dl = \frac{dD}{dt} \int_s dS$$

$$H_\theta 2\pi r = \frac{dD}{dt} \pi r^2$$

$$H_\theta = \frac{dD}{dt} \frac{r}{2}$$

$$D = \epsilon E$$

$$\vec{H} = H_\theta \vec{a}_\theta = \frac{dD}{dt} \frac{r}{2} \vec{a}_\theta = \epsilon \frac{dE}{dt} \frac{r}{2} \vec{a}_\theta$$



(expression for magnetic field at a radial distance r from the axis of the parallel-plate capacitor consisting of circular plates obtained using approach I)

Approach II: $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$ ← $\vec{D} = D \vec{a}_z$



← z component

$$(\nabla \times \vec{H})_z = \frac{\partial D}{\partial t}$$

← expanding left hand side in cylindrical coordinates



$$\frac{1}{r} \left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta} \right) = \frac{\partial D}{\partial t}$$



← $H_\theta \neq 0, H_r = 0$
 $\partial / \partial \theta = 0$ (azimuthal symmetry of the problem)

$$\frac{1}{r} \frac{\partial(rH_\theta)}{\partial r} = \frac{\partial D}{\partial t}$$



← Integrating for uniform D

$$\int d(rH_\theta) = \frac{dD}{dt} \int r dr$$



$$rH_\theta = \frac{dD}{dt} \frac{r^2}{2} + C$$

← $C = \text{integration constant}$

$$rH_{\theta} = \frac{dD}{dt} \frac{r^2}{2} + C$$



$$H_{\theta} = \frac{dD}{dt} \frac{r}{2} + \frac{C}{r}$$



$$H_{\theta} = \frac{dD}{dt} \frac{r}{2}$$



$$\vec{H} = H_{\theta} \vec{a}_{\theta} = \frac{dD}{dt} \frac{r}{2} \vec{a}_{\theta}$$



$$\vec{H} = \epsilon \frac{dE}{dt} \frac{r}{2} \vec{a}_{\theta}$$

For non-zero value of C,

$$H_{\theta} \rightarrow \infty \text{ as } r \rightarrow 0$$

However, the magnetic field blowing up to infinity at the centre of the capacitor is not physically acceptable.

Therefore, for a physically acceptable solution we must put

$$C = 0$$

$$D = \epsilon E$$

(expression for magnetic field at a radial distance r from the axis of the parallel-plate capacitor consisting of circular plates obtained using approach II)

Thus, approaches I and II, which use integral and differential forms of Maxwell's equations respectively, give one and the same expression for magnetic field in the given problem.

Electric scalar potential and magnetic vector potential in time-varying fields

Let us see how the static-field relation

$$\vec{E} = -\nabla V \text{ (static fields)}$$

gets modified in time-varying situations.

Choose arbitrarily a vector \vec{A}
such that

$$\text{it satisfies the relation } \vec{B} = \nabla \times \vec{A}$$

$$\begin{aligned} &\downarrow \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ &\downarrow \\ \nabla \times \vec{E} &= -\frac{\partial(\nabla \times \vec{A})}{\partial t} = -\nabla \times \frac{\partial \vec{A}}{\partial t} \\ &\downarrow \\ \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) &= 0 \end{aligned}$$

partial time derivative and curl
that involves partial derivatives
with the space coordinates and
angles being interchangeable

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 \quad \longleftarrow \text{comparing} \quad \longrightarrow \quad \nabla \times \nabla \psi = 0$$

(vector identity ψ being a scalar quantity)

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = \nabla \psi \quad \xrightarrow{\partial/\partial t = 0 \text{ for static fields}} \quad \vec{E} = \nabla \psi \quad (\text{static fields})$$

$$\vec{E} = -\nabla V \quad (\text{static fields})$$

$$\psi = -V$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \quad (\text{relation between the electric field and two potentials namely scalar electric potential and magnetic vector potential for time-varying situations})$$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \leftarrow \quad \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \text{ (recalled)}$$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\nabla V - \frac{\partial \vec{A}}{\partial t} \right) = \mu_0 \vec{J} - \mu_0 \epsilon_0 \nabla \frac{\partial V}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

Let the arbitrarily chosen magnetic vector potential satisfy the following condition known as Lorentz condition

$$\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} = 0 \text{ (Lorentz condition)}$$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \nabla \left(\mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\mu_0 \epsilon_0 \frac{\partial V}{\partial t} = -\nabla \cdot \vec{A}$$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} + \nabla(\nabla \cdot \vec{A}) - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

(wave equation in magnetic vector potential)

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \text{ (recalled)}$$

$$\leftarrow \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \text{ (recalled)}$$

$$\nabla \cdot \left(-\nabla V - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} = 0 \text{ (Lorentz condition) (recalled)}$$

↓

$$-\nabla^2 V - \frac{\partial}{\partial t} \nabla \cdot \vec{A} = \frac{\rho}{\epsilon_0}$$

$$\leftarrow \nabla \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$$

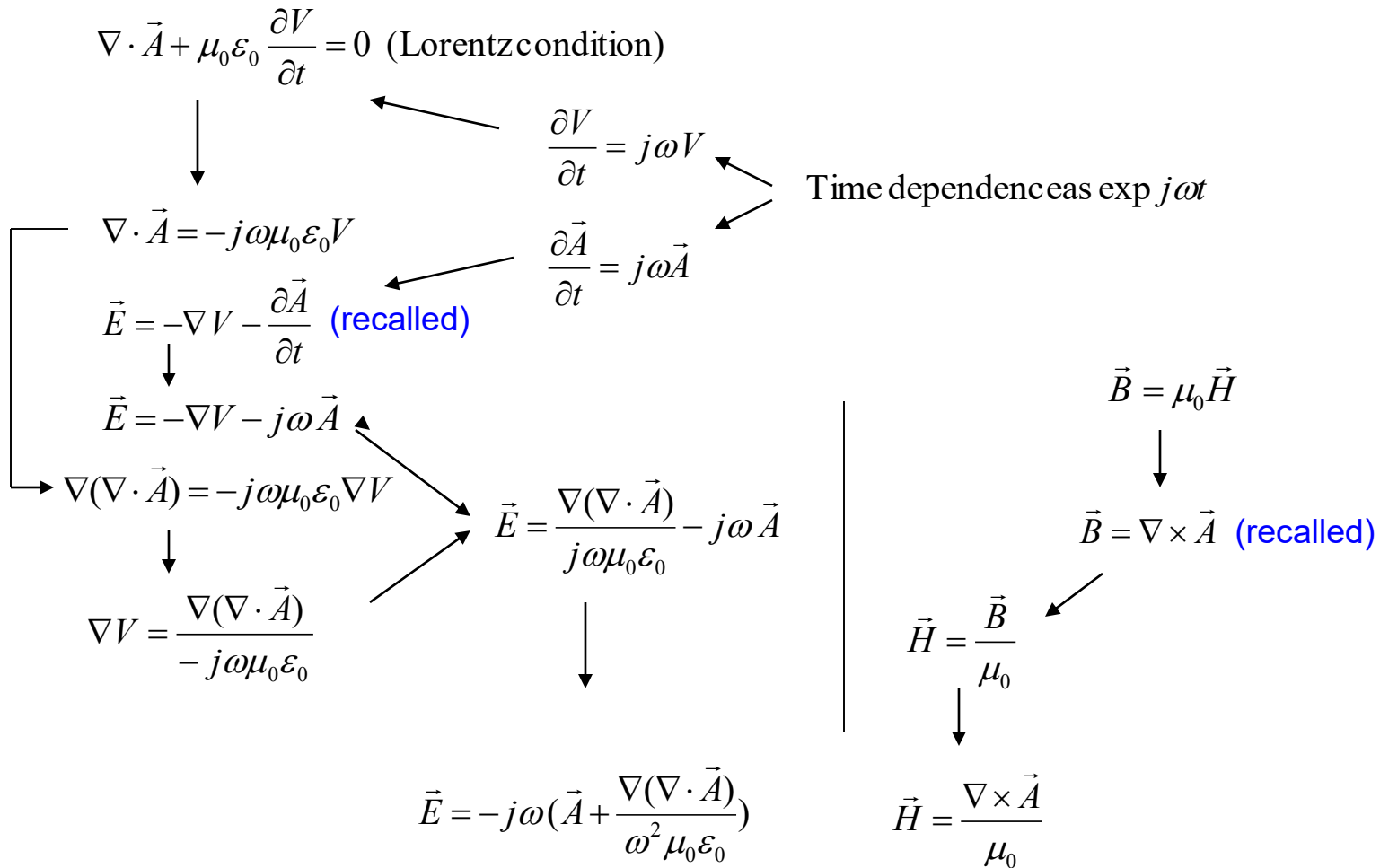
$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

(wave equation in electric scalar potential)

$$\left. \begin{aligned} \nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J} \\ \nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} &= -\frac{\rho}{\epsilon_0} \end{aligned} \right\}$$

(wave equations in electric scalar and magnetic vector potentials put together)

Electric and magnetic fields in terms of magnetic vector potential



(expressions for electric and magnetic fields in terms of magnetic vector potential)

Solutions to wave equations in electric scalar and magnetic vector potential: retarded potentials

Let us take first take wave equation in electric scalar potential

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

↓ ← special case: $\rho = 0, \partial/\partial t \neq 0$

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = 0 \quad (\rho = 0, \partial/\partial t \neq 0)$$

(Special case I: $\rho = 0, \partial/\partial t \neq 0$)

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

↓ ← special case: $\partial/\partial t = 0, \rho \neq 0$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad (\partial/\partial t = 0, \rho \neq 0)$$

(Special case II: $\partial/\partial t = 0, \rho \neq 0$)

The approach followed is to obtain the solutions for the special cases I and II, and then combine them to obtain the solution for the general situation of non-zero values of both volume charge density and time rate of scalar potential: $\partial/\partial t \neq 0, \rho \neq 0$

Special case I:

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = 0 \quad (\rho = 0, \partial / \partial t \neq 0)$$

The solution has well known form representing a wave:

$$V \propto \exp j(\omega t - \beta r)$$

$(\rho = 0, \partial / \partial t \neq 0)$ (SpecialcaseI)

r = distance of the point where the potential is sought from the source of potential

Special case II:

$\beta = \omega / v_p =$ wave propagation constant, v_p being wave phase velocity

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad (\partial / \partial t = 0, \rho \neq 0) \quad (\text{SpecialcaseII})$$

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{r} d\tau \quad (\text{due to a chargedistribution})$$

$(\partial / \partial t = 0, \rho \neq 0)$ (SpecialcaseII)

Solution to be in compliance with the potential due to a static charge distribution which can be written with the help of potential due to a point charge

$$V = \frac{q}{4\pi\epsilon_0} \quad (\text{due to a point charge})$$

$$q = \int_{\tau} \rho d\tau$$

$$V \propto \exp j(\omega t - \beta r)$$

$(\rho = 0, \partial / \partial t \neq 0)$ (Special case I)

$$V = \frac{1}{4\pi\epsilon_0} \int_{\tau} \frac{\rho}{r} d\tau \text{ (due to a chargedistribution)}$$

$(\partial / \partial t = 0, \rho \neq 0)$ (Special case II)



Combined

$$V = \frac{1}{4\pi\epsilon_0} \int_{\tau} \frac{\rho}{r} \exp j(\omega t - \beta r) d\tau$$

$(\partial / \partial t \neq 0, \rho \neq 0)$

(as the solution to wave equation

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} \text{ in scalar electric potential)}$$

Following the same approach we can then obtain

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{\tau} \frac{\vec{J}}{r} \exp j(\omega t - \beta r) d\tau$$

$(\partial / \partial t \neq 0, \vec{J} \neq 0)$

(as the solution to wave equation

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \text{ in vector magnetic potential)}$$

$$\left. \begin{aligned} Jd\tau &= \frac{I}{\alpha} d\tau = \frac{I}{\alpha} \alpha dl = I dl \\ \vec{J} d\tau &= I d\vec{l} \end{aligned} \right\}$$



$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{r} \exp j(\omega t - \beta r) d\tau$$

($\partial/\partial t \neq 0 = 0, \vec{J} \neq 0$)



$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I}{r} \exp j(\omega t - \beta r) d\vec{l}$$

($\partial/\partial t \neq 0 = 0, I \neq 0$)

(solution to wave equation in magnetic vector potential)

(solution to wave equation in magnetic vector potential)

Special case of a steady current: $\exp j(\omega t - \beta r) = 1$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I}{r} d\vec{l} \quad (\text{Biot-Savart's law})$$

$I d\vec{l}$ = current element vector through a filamentary wire of infinitesimal length dl

$d\vec{l}$ = length element vector of magnitude dl the direction of which is the direction of current through the length element

α = cross-sectional area of the filamentary wire

$d\tau = \alpha dl$ = volume element occupied by infinitesimal length of the wire

Retarded potential

$$v_p = \frac{\omega}{\beta}$$
$$V = \frac{1}{4\pi\epsilon_0} \int_{\tau} \frac{\rho}{r} \exp j(\omega t - \beta r) d\tau$$

($\partial/\partial t \neq 0 = 0, \rho \neq 0$)

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{\tau} \frac{I}{r} \exp j(\omega t - \beta r) d\vec{l}$$

($\partial/\partial t \neq 0 = 0, I \neq 0$)

$$V = \frac{1}{4\pi\epsilon_0} \int_{\tau} \frac{\rho}{r} \exp j\omega(t - r/v_p) d\tau$$

($\partial/\partial t \neq 0 = 0, \rho \neq 0$)

(solution to wave equation in electric scalar potential)

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{\tau} \frac{I}{r} \exp j\omega(t - r/v_p) d\vec{l}$$

($\partial/\partial t \neq 0 = 0, I \neq 0$)

(solution to wave equation in magnetic vector potential)

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{r} \exp j\omega(t-r/v_p) d\tau$$

($\partial/\partial t \neq 0 = 0, \rho \neq 0$)

(solution to wave equation in scalar electric potential)

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I}{r} \exp j\omega(t-r/v_p) d\vec{l}$$

($\partial/\partial t \neq 0 = 0, I \neq 0$)

(solution to wave equation in magnetic vector potential)

Retarded scalar electric and vector magnetic potentials: As the source quantities ρ and I vary as $\exp(j\omega t)$ at an instant of time t , the scalar and vector potentials at the observation point at a distance r vary as $\exp(j\omega t')$, where $t' = t - r/v_p$ is a time earlier than t . In other words, these potentials are 'retarded' in time ($t' = t - r/v_p$), meaning thereby that the disturbances in the source, here in the quantities ρ and I , take a time r/v_p to reach the observation point at a distance r while travelling with the velocity v_p , and they manifest themselves in the potential quantities.

Expressions that have practical application in finding field quantities may be put together:

$$\left. \begin{aligned} \vec{A} &= \frac{\mu_0}{4\pi} \int \frac{I}{r} \exp j\omega(t-r/v_p) d\vec{l} \\ \vec{H} &= \frac{\nabla \times \vec{A}}{\mu_0} \\ \vec{E} &= -j\omega \left(\vec{A} + \frac{\nabla(\nabla \cdot \vec{A})}{\omega^2 \mu_0 \epsilon_0} \right) \end{aligned} \right\}$$

Summarising Notes

- ✓ Continuity equation at a point relating the divergence of the current density to the time rate of the variation of volume charge density at the point has been derived starting from the conservation of charge in a region of charge flow.
- ✓ Continuity equation has been used to find the relaxation time of a medium, which measures how long a charge injected into a medium would stay in the bulk of the medium.
- ✓ That a conductor can be charged only at its surface and that a dielectric can be charged throughout its volume can be understood from the fact that the relaxation time of a conductor is very short, while that of a dielectric is very long.
- ✓ Faraday's law for time-varying magnetic fields in both its integral and differential forms has been formulated.
- ✓ Concept of electromagnetic induction has been developed.
- ✓ That electric and magnetic fields are coupled in time-varying phenomena has been appreciated.

✓ Concept of displacement current has been developed with the help of the continuity equation, Poisson's equation and differential form of Ampere's circuital law.

✓ Dielectric medium, even though it is non-conducting, allows the displacement current to pass through it in a time-varying electric field.

✓ Capacitor, though it blocks a direct current, allows a time-varying current to pass through it in the form of displacement current.

✓ Loss tangent of a lossy dielectric in terms its conductivity and permittivity and the frequency of a time-periodic electric field has been derived.

✓ If the region between the conductors of a capacitor is filled up with a lossy dielectric, then the loss tangent of the dielectric measures the extent to which the current through the capacitor fails to lead the voltage across it by the phase angle of $\pi/2$.

- ✓ Maxwell's equations have been obtained in both integral and differential forms.
- ✓ Magnetic vector potential has been introduced defining it such that its curl represents the magnetic flux density.
- ✓ One of Maxwell's equations which relates the electric field with the time rate of variation of magnetic flux density as well as the relation between the magnetic flux density and vector potential has been used to find a relation between the electric field and the electric potential for a time-varying situation.
- ✓ Lorentz condition in terms of the time rate of scalar electric potential and the divergence of magnetic vector potential has been used to derive the wave equations in electric scalar and magnetic vector potentials.
- ✓ Solutions to the wave equations in electric scalar and magnetic vector potentials have been obtained keeping in view their potential applications (for instance, in the analysis of conduction current antennas).

Readers are encouraged to go through Chapter 5 of the book for more topics and more worked-out examples and review questions.