Game Theory with Application in Economics and Finance

Magistère BFA 2 - April 2021

Solution to Final Exam

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Regulating Facebook's planned cryptocurrency (16 pts)

Part A. The United States regulates Facebook's currency in a closed economy (6 pts)

A1. (**3 pts**) The corresponding matrix payoff writes as

		US's choice		
		H	M	L
FB's	S	$(0; a)$	$(0; b)$	$(0; 0)$
choice	C	$(x; c)$	$(y; d)$	$(z; e)$

with x, y, and z, any numbers satisfying:

$$x < 0 < y < z$$

so that FB's profit is decreasing with the level of regulation and is positive (resp. negative) in case of medium and low levels (resp. high level) of regulation. And a, b, c, d, and e, any numbers satisfying:

$$a < b < 0; \text{ and } c > d > e > 0$$

so that US's payoff is positive and increasing (resp. negative and decreasing) with the level of regulation when *Libra* is developed (resp. stopped).

The players' best responses write as: $BR^{US}(S) = \{L\}$, $BR^{US}(C) = \{H\}$, $BR^{FB}(L) = \{C\}$, $BR^{FB}(M) = \{C\}$, $BR^{FB}(H) = \{S\}$. So, there is no pure strategy Nash equilibrium. The set of pure strategy Nash equilibrium is empty.

Clearly, z is FB's highest payoff. So, the outcome (C, L) is Pareto-optimal. Also, c is US's highest payoff. So, the outcome (C, H) is Pareto-optimal as well. The outcome (C, M) is also Pareto-optimal because the only outcome that improves FB's payoff is (C, L) (resp. US's payoff is (C, H)) which would deteriorate US's (resp. FB's) payoff. Any other outcomes are Pareto-dominated by one of these three outcomes. Therefore, the set of Pareto-efficient outcomes is $\{(C, L); (C, M); (C, H)\}$.

A2. (**3 pts**) The corresponding matrix payoff writes as

		US's choice	
		M	L
FB's	S	$(0; b)$	$(0; 0)$
choice	C	$(y; d)$	$(z; e)$

so that US's payoff is increasing (resp. decreasing) with the level of regulation when *Libra* is developed (resp. stopped).

Clearly, C is now FB's strictly dominant strategy. Now that the action H is no more available to US, $BR^{US}(C) = \{M\}$. The set of pure strategy Nash equilibrium is the singleton $\{(C, M)\}$ with the interpretation that at equilibrium *Libra* is developed under a medium level of regulation.

Clearly, z is FB's highest payoff. So, the outcome (C, L) is Pareto-optimal. Also, d is US's highest payoff. So, the outcome (C, M) is Pareto-optimal as well. Any other outcomes are Pareto-dominated by one of these two outcomes. Therefore, the set of Pareto-efficient outcomes is $\{(C, L); (C, M)\}$.

Part B. The United States regulates Facebook's currency with China as a competitor (10 pts)

B1. (**2 pts**) The corresponding matrix payoff writes as

CH's choice

		US's choice	
		M	L
FB's	S	$(0; b; \alpha)$	$(0; 0; \alpha)$
choice	C	$(y^A; d^A; \beta)$	$(z^A; e^A; \gamma)$

$\xleftarrow{A}$ $\xrightarrow{F}$

		US's choice	
		M	L
FB's	S	$(0; b; \alpha)$	$(0; 0; \alpha)$
choice	C	$(y^F; d^F; \delta)$	$(z^F; e^F; \epsilon)$

with y^A, z^A, y^F and z^F, any numbers satisfying:

$$y^F < y < y^A,\ z^F < z < z^A,\ 0 < y^A < z^A \text{ and } y^F < 0 < z^F$$

and b, d^A, e^A, d^F, and e^F, any numbers satisfying:

$$d^F < d < d^A,\ e^F < e < e^A,\ b < 0,\ d^A > e^A > 0 \text{ and } d^F > e^F$$

so that under *Libra* development, FB and US's payoffs are higher (resp. lower) than before when CH accommodate (resp. fight), and in case of a Chinese fight FB's profit would become negative under a medium level of regulation. Also, α, β, γ, δ, and ϵ are any numbers satisfying:

$$\alpha > \delta > \varepsilon > \beta > \gamma$$

so that when *Libra* is not developed, CH's payoff is maximal and does not depend on US's regulation (i.e., $\alpha = \max\{\alpha; \delta; \varepsilon; \beta; \gamma\}$). Otherwise (under *Libra* development), CH is in favor of a most regulated version of *Libra* (i.e., $\beta > \gamma$ and $\delta > \epsilon$). CH prefers to fight a low regulated *Libra* than to accommodate a medium regulated American crypto-currency (i.e., $\epsilon > \beta$).

B2. (**1 pt**) The (simultaneous) subgame where CH accommodates writes as

		US's choice	
		M	L
FB's	S	$(0; b; \alpha)$	$(0; 0; \alpha)$
choice	C	$(y^A; d^A; \beta)$	$(z^A; e^A; \gamma)$

Clearly, C is FB's strictly dominant strategy. US' best responses write as: $BR^{US}(S, A) = \{L\}$ and $BR^{US}(C, A) = \{M\}$. So, there is a unique pure strategy Nash equilibrium. The set of pure strategy Nash equilibrium is the singleton $\{(C, M)\}$.

B3. (**3 pts**) The (simultaneous) subgame where CH fights writes as

		US's choice	
		M	L
FB's	S	$(0; b; \alpha)$	$(0; 0; \alpha)$
choice	C	$(y^F; d^F; \delta)$	$(z^F; e^F; \epsilon)$

The players' best responses write as: $BR^{US}(S, A) = \{L\}$, $BR^{US}(C, A) = \{M\}$, $BR^{FB}(L, A) = \{C\}$, $BR^{FB}(M, A) = \{S\}$. So, there is no pure strategy Nash equilibrium.
Applying the indifference property we can characterize the mixed strategy equilibrium. Let p (resp. q) denotes the probability according to which FB (resp. US) stops the development of *Libra* (resp. applies a medium level of regulation). The pair (p, q) solves the system:

$$\begin{cases} p \times b + (1-p) \times d^F = p \times 0 + (1-p) \times e^F \\ q \times 0 + (1-q) \times 0 = q \times y^F + (1-q) \times z^F \end{cases}$$

which is equivalent to

$$\begin{cases} p = \frac{d^F - e^F}{d^F - e^F - b} \\ q = \frac{z^F}{z^F - y^F} \end{cases}$$

The set of pure strategy Nash equilibrium is the singleton $\{(p^*, q^*) = \left(\frac{d^F - e^F}{d^F - e^F - b}, \frac{z^F}{z^F - y^F}\right)\}$.

The likelihood p^* is increasing with both b and e^F (since $\frac{\partial p^*}{\partial b} = \frac{d^F - e^F}{(d^F - e^F - b)^2} > 0$, and $\frac{\partial p^*}{\partial e^F} = \frac{b}{(d^F - e^F - b)^2} > 0$), and decreasing with d^F (since $\frac{\partial p^*}{\partial d^F} = -\frac{\partial p^*}{\partial e^F} < 0$). So, the likelihood that the project stops at equilibrium increases with US's payoff associated to an ongoing project regulated at a low level (e^F) and an aborted project that would have been regulated at a medium level (b), and decreases with US's payoff under an ongoing highly regulated project (d^F). The likelihood q^* is increasing with both y^F and z^F (since $\frac{\partial q^*}{\partial y^F} = \frac{z^F}{(z^F - y^F)^2} > 0$ and $\frac{\partial q^*}{\partial z^F} = \frac{-y^F}{(z^F - y^F)^2} > 0$). So, the higher FB's payoff associated to an ongoing project (either regulated at a medium or low level), the more likely US regulate at a medium level at equilibrium.

B4. (**2 pts**) The players' best responses write as:

$$\begin{aligned} BR^{FB}(.,A) &= \{C\};\ BR^{FB}(M,F) = \{S\};\text{ and } BR^{FB}(L,F) = \{C\} \\ BR^{US}(S,.) &= \{L\};\text{ and } BR^{US}(C,.) = \{M\} \\ BR^{CH}(S,.) &= \{A,F\};\text{ and } BR^{CH}(C,.) = \{F\} \end{aligned}$$

So there is no pure strategy equilibrium.
From

$$\delta > \beta \text{ and } \varepsilon > \gamma$$

F is CH's weakly dominant strategy and is CH's unique best response when C is played by FB with a strictly positive probability. Since there is no equilibrium sustained by S (indeed, $BR^{US}(S,.) = \{L\} \notin BR^{FB}(L,.)$), there is then a unique equilibrium. It corresponds to the previous mixed strategy equilibrium where CH plays F and FB and US play according to (p^*, q^*). The set of strategy Nash equilibria is a singleton: $\{(p^*, q^*, F)\}$.

B5. (**1 pt**) The resulting equilibrium expected payoffs are as follows. When it fights, CH's expected payoff writes as

$$\alpha p^* q^* + \alpha p^* (1 - q^*) + \delta (1 - p^*) q^* + \varepsilon (1 - p^*)(1 - q^*)$$

that is

$$\alpha p^* + (1 - p^*)(\delta q^* + \varepsilon (1 - q^*))$$

FB's expected payoffs writes as

$$\begin{aligned} & y^F (1 - p^*) q^* + z^F (1 - p^*)(1 - q^*) \\ = \ & \frac{b}{b + e^F - d^F} \left(y^F \frac{z^F}{z^F - y^F} + z^F \frac{-y^F}{z^F - y^F} \right) = 0 \end{aligned}$$

US's expected payoffs writes as

$$\begin{aligned} & b p^* q^* + d^F (1 - p^*) q^* + e^F (1 - p^*)(1 - q^*) \\ = \ & b \frac{e^F - d^F}{b + e^F - d^F} \frac{z^F}{z^F - y^F} + d^F \frac{b}{b + e^F - d^F} \frac{z^F}{z^F - y^F} + e^F \frac{b}{b + e^F - d^F} \frac{-y^F}{z^F - y^F} \\ = \ & \frac{b e^F (z^F - y^F)}{(b + e^F - d^F)(z^F - y^F)} = \frac{b e^F}{(b + e^F - d^F)} \end{aligned}$$

and CH's expected payoffs writes as

$$\begin{aligned} & \alpha p^* + (1-p^*)\left(\delta q^* + \varepsilon\left(1-q^*\right)\right) \\ = \; & \alpha \frac{e^F - d^F}{b + e^F - d^F} + \frac{b}{b + e^F - d^F}\left(\delta \frac{z^F}{z^F - y^F} + \varepsilon \frac{-y^F}{z^F - y^F}\right) \\ = \; & \frac{\alpha\left(e^F - d^F\right) + b\left(\delta z^F - \varepsilon y^F\right)}{\left(b + e^F - d^F\right)\left(z^F - y^F\right)} \end{aligned}$$

B6. (**1 pt**) The equilibrium is Pareto optimal. Indeed, CH's expected payoffs is increasing in p^* while US's expected payoffs is decreasing in p^*. So, any change in p^* necessarily decrease at least one player's payoffs. A similar argument can be used with respect to q^*, observing that for any fixed probability p, US's (resp. FB's) expected payoffs is increasing (resp. decreasing) with q^*. Finally, for any fixed pair of probabilities (p, q), CH would be worse off by fighting with lower probability.

Dilemme du prisonnier répété (4 pts)

1. (**1 pt**) La stratégie « grim trigger » consiste ici pour le joueur i à jouer :
   - c_i à la période $t = 1$; puis
   - à la période $t > 1$, jouer c_i si (c_1,c_2) a été joué jusqu'à la période $(t-1)$, et jouer t_i sinon.

2. (**1 pt**) Lorsque le jeu est répété de manière infinie, le paiement espéré le long du chemin de la coopération s'écrit:

$$3 \sum_{t=0}^{+\infty} \delta^t = \frac{3}{1-\delta}.$$

Le paiement espéré le plus élevé de la déviation à la période k, s'écrit :

$$3 \times \sum_{t=0}^{k-1} \delta^t + (4+\alpha)\delta^k + 1 \times \sum_{t=k+1}^{+\infty} \delta^t = \frac{3 \times (1-\delta^k) + (4+\alpha) \times (1-\delta)\delta^k + 1 \times \delta^{k+1}}{1-\delta}$$

(**1 pt**) La première expression est supérieure à la seconde si et seulement si

$$3 \times \delta^k \geq (4+\alpha) \times \delta^k + (1-(4+\alpha)) \times \delta^{k+1}$$

C'est-à-dire lorsque

$$\delta^k(1+\alpha) \leq \delta^{k+1}(3+\alpha)$$

et donc

$$\delta \geq \frac{1+\alpha}{3+\alpha} := \bar{\delta}(\alpha).$$

3. (**1 pt**) Clairement,

$$\frac{\partial \bar{\delta}(\alpha)}{\partial \alpha} = \frac{2}{(3+\alpha)^2} > 0.$$

Le seuil $\overline{\delta}$ est donc croissant de α. Ce résultat correspond à l'intuition selon laquelle plus la déviation unilatérale par rapport à la coopération mutuelle est profitable, c'est-à-dire plus α est élevé, et plus les joueurs ont besoin de valoriser le futur (δ élevé) pour que la perspective d'une punition future les incitent à ne pas trahir la coopération actuelle.