The Physics of Stacking

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In this experiment, we will explore a problem popularly known as "The Leaning Tower of Lire" in mathematical literature [1]. This involves creating a mathematical model in order to determine the arrangement of stacking blocks in a manner that maximizes the overhang—which is essentially a measure of how much a stack *leans* outward.

This seemingly simple problem will help you realize the challenge of attaining stability in various designs. Popularly, this is used in modelling bridges and arches. You will see how the stability of a structure is governed by carefully tinkering with the center of mass which you will explore hands-on in this experiment. You will also make use of mathematical induction and realize how it is a powerful tool in arriving at a mathematical model that assists us in solving physical problems.

KEYWORDS

Overhang · Harmonic Series · Mathematical Induction · Center of Mass · Projection · Partial Sum ·

1 Objectives

In this experiment, we will,

1. revisit the concept of the center of mass,
2. build a mathematical model for maximizing the overhang, and see how we build a mathematical model that explains a physical observation.

2 Theoretical Background

2.1 Center of Mass

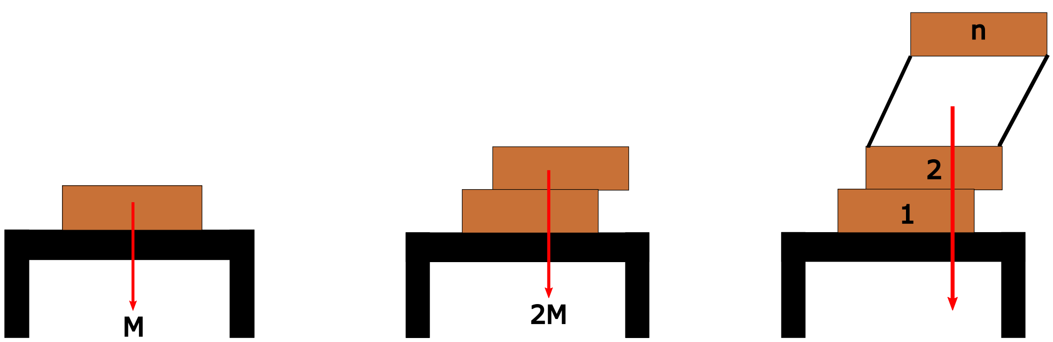
The concept of the center of mass assumes that the mass of an object can be taken to be concentrated at one point. We start by considering a system of n particles for which we can define the center of mass to be the weighted average of the different masses. In one dimension, this leads to:

$$X_c = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}, \tag{1}$$

where m_i and x_i are the mass and the position of the i'th object while X_c is the center of mass of the n objects.

Similar expressions can be written down for Y_c and Z_c.

2.2 Demonstrating changes in the center of mass



(a) One block. (b) Two blocks. (c) n blocks.

Figure 1: Schematic diagram showing how the center of mass shifts as blocks are stacked on top of one another.

We will now look at how building a stack of objects that are not aligned shifts their collective center of mass. Figure(1) shows how the center of mass shifts as the blocks are placed on top of one another. For one object, the line of action of the weight is shown in Figure (1a).

In Figure (1b), the addition of another object changes the centre of mass such that it now lies at the geometrical centre of the new composite object.

If we keep on adding more and more identical objects, the center of mass will shift according to Equation (1). This scenario is shown in Figure (1c).

Explain: How would you arrange the blocks in order to prevent them from toppling over?

2.3 Constructing the Harmonic Overhang

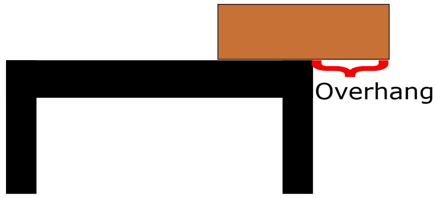
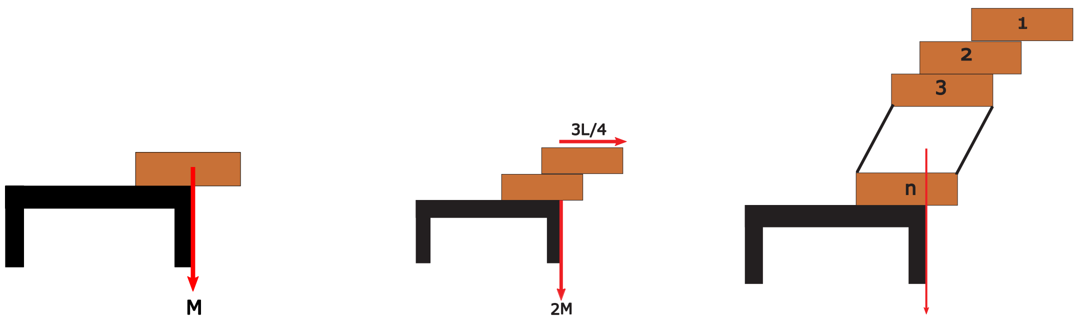


Figure 2: Defining the Overhang.

As shown in Figure (2), the overhang is simply a measure of how far off an object protrudes from the edge of a surface such as a table.



(a) One block. (b) Two blocks. (c) n blocks.

Figure 3: Showing the maximum overhang for one, two, and n blocks.

We will explore how to maximise this protrusion for a system of n objects. Maximizing the overhang requires that the collective center of mass of the blocks should coincide with the edge of the table. This is shown by the growing number of blocks in Figure (3).

We denote the overhang by $S(k)$ where k denotes the number of blocks. Suppose the length of each block is l. As such, $S(1) = \frac{l}{2}$.

Exercise: By taking the center of mass at the edge of the table, show that:

$$S(2) = (\frac{l}{2} + \frac{l}{4}), \tag{2}$$

as shown in Figure (3b). (Hint: First, arrange the blocks such that the bottom block is not projected outwards and only the second block is projected by $l/2$. Find out the center of mass of this configuration and see how much you can displace the configuration further outwards from the table such that the center of mass shifts to the edge. This will give you $S(2)$.)

Exercise: For three blocks, show that:

$$S(3) = (\frac{l}{2} + \frac{l}{4} + \frac{l}{6}). \tag{3}$$

As shown in Figure (3c), we now further build this configuration and introduce a stack of n objects whose combined center of mass is at the edge of the table.

Exercise: Now, by considering $S(1)$, $S(2)$, $S(3)$, $S(4)$, $S(5)$, and so on, write down the mathematical pattern for n blocks.

Exercise: Show that this pattern can be written as:

$$S_n = \frac{1}{2}\sum_{k=1}^{n} \frac{l}{k}. \tag{4}$$

This sum is the famous harmonic sum and describes the maximal overhang followed by a stack of n objects, each of length l. With this configuration, the center of mass of the configuration is at the edge of the table and hence, it is a stable configuration.

2.3.1 Mathematical Induction

Mathematical induction is a technique of mathematical proof that proves that a property $P(n)$ holds for every natural number n, i.e. for $n = 1, 2, 3$, and so on.

The method of induction requires two cases to be established. The first case, called the "base case" proves that the property holds for the number 1. The second case, called the "induction step", proves that if the property holds for one natural number n, then it also holds for the next natural number in the sequence, i.e. $n + 1$. These two steps establish the property for $P(n)$.

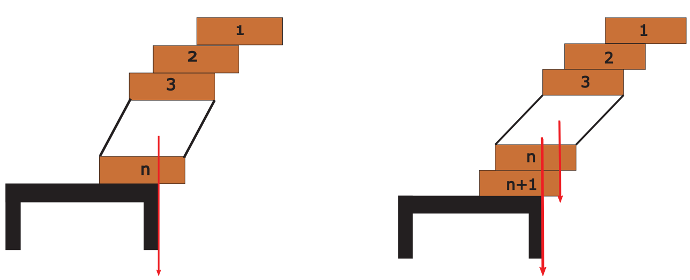
In order to make use of mathematical induction, we now introduce another block that is to be placed below these n objects. Yet again, we can only do so by ensuring that the new bottom most block is shifted to the left as shown in Figure (4b).

Exercise: Prove Equation (4) using mathematical induction.

Exercise: More so, find the change in the overhang $\Delta S(n)$ in going from n to $n + 1$ blocks. This is given as:

$$\Delta S(n) = S(n + 1) - S(n). \tag{5}$$

$S(n)$ and $S(n+1)$ are demonstrated in Figure (4a) and Figure (4b) respectively. The arrows in Figure(4b) show the old and new centers of mass.



(a) n blocks. (b) $(n + 1)$ blocks.

Figure 4: Schematic showing overhangs of n and $(n + 1)$ blocks.

3 The Experiment

You have been provided with a set of identical blocks.

1. Measure the length l of each block.
2. Using Equation (4) and the length found above, calculate the overhang for values of $n = 1, 2, 3$, and so on. It is expected that you will be able to construct the overhang for about a dozen blocks. Make a table of values that includes the overh. Your table should include the overhang due to n blocks and the projection caused by the n'th block
3. Now, using the blocks provided to you, demonstrate the overhang for values of $n = 1, 2, 3$, and so on. For each value of n, find the value of $S(n)$ that results in a stable structure. Make a table of values.

4 Analyzing the Data

1. Using MATLAB, write a program that generates the values for the overhang due to $n = 1, 2, \ldots$ blocks. This program should also generate values for the projections due to each block. Using a suitable choice of axes, plot a graph that shows how the overhang grows. Also, plot a graph that shows how the projection of the n'th block changes. Explain: Comment on the behavior shown by the graphs.
2. Using the table of values for the experimental values, plot graphs for overhangs and projections showing error bars.
3. Compare plots of your experimental data for $S(n)$ and $\Delta S(n)$ with the theoretical predictions. Could you explain the reasons for any discrepancy you observe?
4. Use your MATLAB code to generate values for larger values of n. What happens as n approaches infinity? What does this tell us about using the harmonic sum to model the overhang?
5. How would you make this experiment more accurate?

Explore: Come up with a different model for building a stack and measure its overhang. How would you optimize the overhang in this particular model?

References

[1] Johnson, Paul B. (April 1955). *"Leaning Tower of Lire"*. American Journal of Physics. 23 (4): 240–240.

[2] Paul J. Nahin, *In Praise of Simple Physics: The Science and Mathematics behind Everyday Questions*, Princeton University Press, 2016.