1

Physics and Fourier transforms

1.1 The qualitative approach

Ninety percent of all physics is concerned with vibrations and waves of one sort or another. The same basic thread runs through most branches of physical science, from acoustics through engineering, fluid mechanics, optics, electromagnetic theory and X-rays to quantum mechanics and information theory. It is closely bound to the idea of a *signal* and its *spectrum*. To take a simple example: imagine an experiment in which a musician plays a steady note on a trumpet or a violin, and a microphone produces a voltage proportional to the instantaneous air pressure. An oscilloscope will display a graph of pressure against time, $F(t)$, which is periodic. The reciprocal of the period is the frequency of the note, 440 Hz, say, for a well-tempered middle A – the tuning-up frequency for an orchestra.

The waveform is not a pure sinusoid, and it would be boring and colourless if it were. It contains 'harmonics' or 'overtones': multiples of the fundamental frequency, with various amplitudes and in various phases,[1] depending on the timbre of the note, the type of instrument being played and on the player. The waveform can be *analysed* to find the amplitudes of the overtones, and a list can be made of the amplitudes and phases of the sinusoids which it comprises. Alternatively a graph, $A(\nu)$, can be plotted (the sound-spectrum) of the amplitudes against frequency (Fig. 1.1).

$A(\nu)$ **is the Fourier transform of** $F(t)$.

Actually it is the *modular* transform, but at this stage that is a detail.

Suppose that the sound is not periodic – a squawk, a drumbeat or a crash instead of a pure note. Then to describe it requires not just a set of overtones

[1] 'Phase' here is an angle, used to define the 'retardation' of one wave or vibration with respect to another. One wavelength retardation, for example, is equivalent to a phase difference of 2π. Each harmonic will have its own phase, ϕ_m, indicating its position within the period.

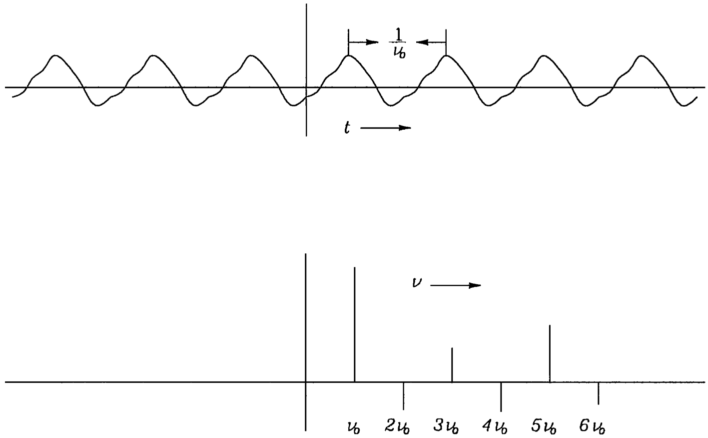

Fig. 1.1. The spectrum of a steady note: fundamental and overtones.

with their amplitudes, but a continuous range of frequencies, each present in an infinitesimal amount. The two curves would then look like Fig. 1.2.

The uses of a Fourier transform can be imagined: the identification of a valuable violin; the analysis of the sound of an aero-engine to detect a faulty gear-wheel; of an electrocardiogram to detect a heart defect; of the light curve of a periodic variable star to determine the underlying physical causes of the variation: all these are current applications of Fourier transforms.

1.2 Fourier series

For a steady note the description requires only the fundamental frequency, its amplitude and the amplitudes of its harmonics. A discrete sum is sufficient. We could write

$$F(t) = a_0 + a_1 \cos(2\pi \nu_0 t) + b_1 \sin(2\pi \nu_0 t) + a_2 \cos(4\pi \nu_0 t) \\ + b_2 \sin(4\pi \nu_0 t) + a_3 \cos(6\pi \nu_0 t) + \cdots ,$$

where ν_0 is the fundamental frequency of the note. Sines as well as cosines are required because the harmonics are not necessarily 'in step' (i.e. 'in phase') with the fundamental or with each other.

More formally:

$$F(t) = \sum_{n=-\infty}^{\infty} a_n \cos(2\pi n \nu_0 t) + b_n \sin(2\pi n \nu_0 t) \tag{1.1}$$

and the sum is taken from $-\infty$ to ∞ for the sake of mathematical symmetry.

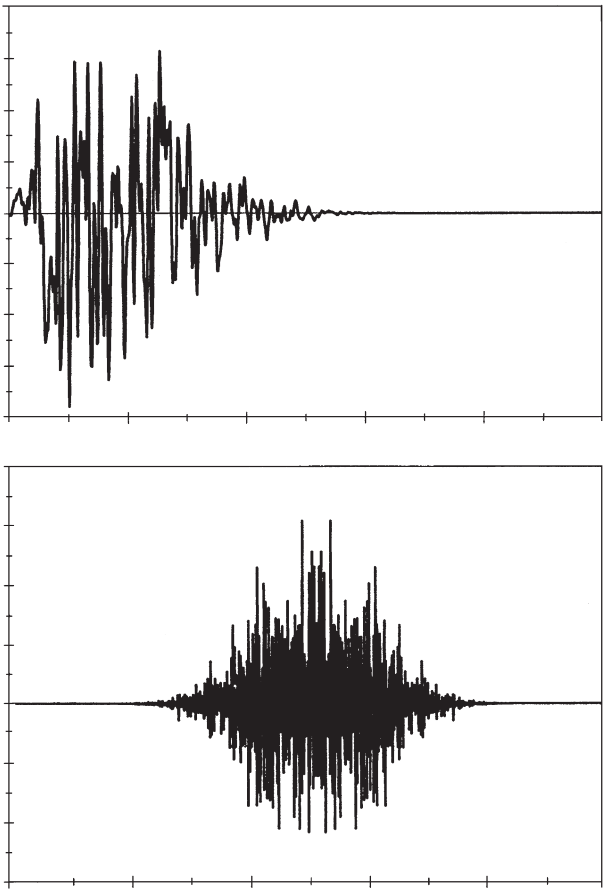

Fig. 1.2. The spectrum of a crash: all frequencies are present.

This process of constructing a waveform by adding together a fundamental frequency and overtones or harmonics of various amplitudes is called Fourier synthesis.

There are alternative ways of writing this expression: since $\cos x = \cos(-x)$ and $\sin x = -\sin(-x)$ we can write

$$F(t) = A_0/2 + \sum_{n=1}^{\infty} A_n \cos(2\pi n \nu_0 t) + B_n \sin(2\pi n \nu_0 t) \tag{1.2}$$

and the two expressions are identical, provided that we set $A_n = a_{-n} + a_n$ and $B_n = b_n - b_{-n}$. A_0 is divided by two to avoid counting it twice: as it is, A_0 can be found by the same formula that will be used to find all the A_n's.

Mathematicians and some theoretical physicists write the expression as

$$F(t) = A_0/2 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)$$

and there are entirely practical reasons, which are discussed later, for *not* writing it this way.

1.3 The amplitudes of the harmonics

The alternative process – of extracting from the signal the various frequencies and amplitudes that are present – is called *Fourier analysis* and is much more important in its practical physical applications. In physics, we usually find the curve $F(t)$ experimentally and we want to know the values of the amplitudes A_m and B_m for as many values of m as necessary. To find the values of these amplitudes, we use the *orthogonality* property of sines and cosines. This property is that, if you take a sine and a cosine, or two sines or two cosines, each a multiple of some fundamental frequency, multiply them together and integrate the product over one period of that frequency, the result is always zero except in special cases.

If $P = 1/\nu_0$ is one period, then

$$\int_{t=0}^{P} \cos(2\pi n\nu_0 t) \cdot \cos(2\pi m\nu_0 t) dt = 0$$

and

$$\int_{t=0}^{P} \sin(2\pi n\nu_0 t) \cdot \sin(2\pi m\nu_0 t) dt = 0$$

unless $m = \pm n$, and

$$\int_{t=0}^{P} \sin(2\pi n\nu_0 t) \cdot \cos(2\pi m\nu_0 t) dt = 0$$

always.

The first two integrals are both equal to $1/(2\nu_0)$ if $m = n$.

We multiply the expression (1.2) for $F(t)$ by $\sin(2\pi m\nu_0 t)$ and the product is integrated over one period, P:

$$\int_{t=0}^{P} F(t)\sin(2\pi m\nu_0 t) dt = \frac{A_0}{2}\int_{t=0}^{P} \sin(2\pi m\nu_0 t) dt + \int_{t=0}^{P} \sum_{n=1}^{\infty} \{A_n \cos(2\pi n\nu_0 t) + B_n \sin(2\pi n\nu_0 t)\}\sin(2\pi m\nu_0 t) dt \quad (1.3)$$

and all the terms of the sum vanish on integration except

$$\int_0^P B_m \sin^2(2\pi m\nu_0 t)dt = B_m \int_0^P \sin^2(2\pi m\nu_0 t)dt$$
$$= B_m/(2\nu_0) = B_m P/2$$

so that

$$B_m = (2/P)\int_0^P F(t)\sin(2\pi m\nu_0 t)dt \tag{1.4}$$

and, provided that $F(t)$ is known in the interval $0 \rightarrow P$, the coefficient B_m can be found. If an analytic expression for $F(t)$ is known, the integral can often be done. On the other hand, if $F(t)$ has been found experimentally, a computer is needed to do the integrations.

The corresponding formula for A_m is

$$A_m = (2/P)\int_0^P F(t)\cos(2\pi m\nu_0 t)dt. \tag{1.5}$$

The integral can start anywhere, not necessarily at $t = 0$, so long as it extends over one period.

Example: Suppose that $F(t)$ is a square-wave of period $1/\nu_0$, so that $F(t) = h$ for $t = -b/2 \rightarrow b/2$ and 0 during the rest of the period, as in Fig. 1.3. Then

$$A_m = 2\nu_0 \int_{-1/(2\nu_0)}^{1/(2\nu_0)} F(t)\cos(2\pi m\nu_0 t)dt$$
$$= 2h\nu_0 \int_{-b/2}^{b/2} \cos(2\pi m\nu_0 t)dt$$

and the new limits cover only that part of the cycle where $F(t)$ is different from zero.

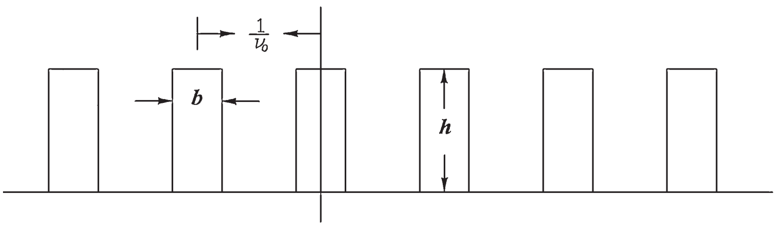

Fig. 1.3. A rectangular wave of period $1/\nu_0$ and pulse-width b.

If we integrate and put in the limits:

$$\begin{aligned}A_m &= \frac{2h\nu_0}{2\pi m\nu_0}\{\sin(\pi m\nu_0 b) - \sin(-\pi m\nu_0 b)\}\\ &= \frac{2h}{\pi m}\sin(\pi m\nu_0 b)\\ &= 2h\nu_0 b\{\sin(\pi\nu_0 mb)/(\pi\nu_0 mb)\}.\end{aligned}$$

All the B_n's are zero because of the symmetry of the function – we took the origin to be at the centre of one of the pulses.

The original function of time can be written

$$F(t) = h\nu_0 b + 2h\nu_0 b\sum_{m=1}^{\infty}\{\sin(\pi\nu_0 mb)/(\pi\nu_0 mb)\}\cos(2\pi m\nu_0 t) \qquad (1.6)$$

or, alternatively,

$$F(t) = \frac{hb}{P} + \frac{2hb}{P}\sum_{m=1}^{\infty}\{\sin(\pi\nu_0 mb)/(\pi\nu_0 mb)\}\cos(2\pi m\nu_0 t). \qquad (1.7)$$

Notice that the first term, $A_0/2$, is the *average* height of the function – the area under the top-hat divided by the period; and that the function $\sin(x)/x$, called 'sinc(x)', which will be described in detail later, has the value unity at $x = 0$, as can be shown using de l'Hôpital's rule.[2]

There are other ways of writing the Fourier series. It is convenient occasionally, though less often, to write $A_m = R_m\cos\phi_m$ and $B_m = R_m\sin\phi_m$, so that equation (1.2) becomes

$$F(t) = \frac{A_0}{2} + \sum_{m=1}^{\infty} R_m\cos(2\pi m\nu_0 t + \phi_m) \qquad (1.8)$$

and R_m and ϕ_m are the amplitude and phase of the mth harmonic. A single sinusoid then replaces each sine and cosine, and the two quantities needed to define each harmonic are these amplitudes and phases in place of the previous A_m and B_m coefficients. In practice it is usually the amplitude, R_m, which is important, since the energy in an oscillator is proportional to the square of the amplitude of oscillation, and $|R_m|^2$ gives a measure of the power contained in each harmonic of a wave. 'Phase' is a simple and important idea. Two wave trains are 'in phase' if wave crests arrive at a certain point together. They are 'out of phase' if a trough from one arrives at the same time as the crest of the other. (Alternatively, they have 180° phase difference.) In Fig. 1.4 there are two

[2] De l'Hôpital's rule is that, if $f(x) \to 0$ as $x \to 0$ and $\phi(x) \to 0$ as $x \to 0$, the ratio $f(x)/\phi(x)$ is indeterminate, but is equal to the ratio $(df/dx)/(d\phi/dx)$ as $x \to 0$.

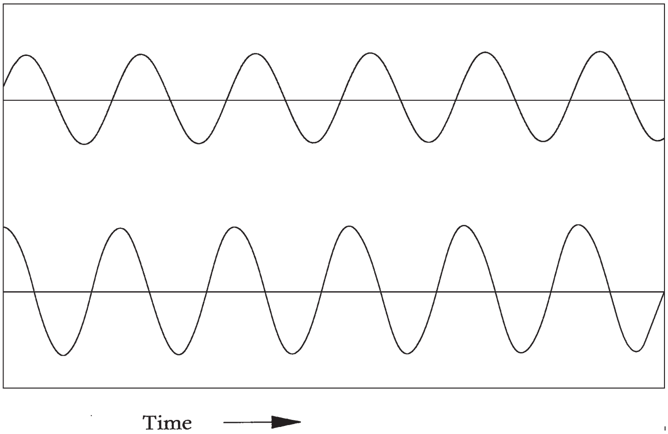

Fig. 1.4. Two wave trains with the same period but different amplitudes and phases. The upper has 0.7 times the amplitude of the lower and there is a phase-difference of 70°.

wave trains. The upper has 0.7 times the amplitude of the other and it *lags* (not *leads*, as it appears to do) the lower by 70°. This is because the horizontal axis of the graph is time, and the vertical axis measures the amplitude at a fixed point as it varies with time. Wave crests from the lower wave train arrive earlier than those from the upper. The important thing is that the 'phase-difference' between the two is 70°.

The most common way of writing the series expansion is with complex exponentials instead of trigonometrical functions. This is because the algebra of complex exponentials is easier to manipulate. The two ways are linked, of course, by de Moivre's theorem. We can write

$$F(t) = \sum_{-\infty}^{\infty} C_m e^{2\pi i m \nu_0 t},$$

where the coefficients C_m are now complex numbers in general and $C_m = C^*_{-m}$. (The exact relationship is given in detail in Appendix A.3.) The coefficients A_m, B_m and C_m are obtained from the *inversion formulae:*

$$A_m = 2\nu_0 \int_0^{1/\nu_0} F(t)\cos(2\pi m \nu_0 t)dt,$$

$$B_m = 2\nu_0 \int_0^{1/\nu_0} F(t)\sin(2\pi m \nu_0 t)dt,$$

$$C_m = 2\nu_0 \int_0^{1/\nu_0} F(t)e^{-2\pi m \nu_0 t}\, dt$$

(the minus sign in the exponent is important) or, if ω_0 has been used instead of ν_0 ($\nu_0 = \omega_0/(2\pi)$), then

$$A_m = (\omega_0/\pi)\int_0^{2\pi/\omega_0} F(t)\cos(m\omega_0 t)dt,$$
$$B_m = (\omega_0/\pi)\int_0^{2\pi/\omega_0} F(t)\sin(m\omega_0 t)dt,$$
$$C_m = (2\omega_0/\pi)\int_0^{2\pi/\omega_0} F(t)e^{-im\omega_0 t}\,dt.$$

The useful mnemonic form to remember for finding the coefficients in a Fourier series is

$$A_m = \frac{2}{\text{period}}\int_{\text{one period}} F(t)\cos\left\{\frac{2\pi mt}{\text{period}}\right\}dt, \tag{1.9}$$
$$B_m = \frac{2}{\text{period}}\int_{\text{one period}} F(t)\sin\left\{\frac{2\pi mt}{\text{period}}\right\}dt \tag{1.10}$$

and remember that the integral can be taken from any starting point, a, provided that it extends over one period to an upper limit $a + P$. The integral can be split into as many subdivisions as needed if, for example, $F(t)$ has different analytic forms in different parts of the period.

1.4 Fourier transforms

Whether $F(t)$ is periodic or not, a complete description of $F(t)$ can be given using sines and cosines. If $F(t)$ is not periodic it requires all frequencies to be present if it is to be synthesized. A non-periodic function may be thought of as a limiting case of a periodic one, where the period tends to infinity, and consequently the fundamental frequency tends to zero. The harmonics are more and more closely spaced and in the limit there is a continuum of harmonics, each one of infinitesimal amplitude, $a(\nu)d\nu$, for example. The summation sign is replaced by an integral sign and we find that

$$F(t) = \int_{-\infty}^{\infty} a(\nu)d\nu\cos(2\pi\nu t) + \int_{-\infty}^{\infty} b(\nu)d\nu\sin(2\pi\nu t) \tag{1.11}$$

or, equivalently,

$$F(t) = \int_{-\infty}^{\infty} r(\nu)\cos(2\pi\nu t + \phi(\nu))d\nu \tag{1.12}$$

or, again,

$$F(t) = \int_{-\infty}^{\infty} \Phi(\nu)e^{2\pi i\nu t}\,d\nu. \tag{1.13}$$

If $F(t)$ is real, that is to say, if the insertion of any value of t into $F(t)$ yields a real number, then $a(\nu)$ and $b(\nu)$ are real too. However, $\Phi(\nu)$ may be complex and indeed will be if $F(t)$ is asymmetrical so that $F(t) \neq F(-t)$. This can sometimes cause complications, and these are dealt with in Chapter 8: but $F(t)$ is often symmetrical and then $\Phi(\nu)$ is real and $F(t)$ comprises only cosines. We *could* then write

$$F(t) = \int_{-\infty}^{\infty} \Phi(\nu)\cos(2\pi\nu t)d\nu$$

but, because complex exponentials are easier to manipulate, we take equation (1.13) above as the standard form. Nevertheless, for many practical purposes only real and symmetrical functions $F(t)$ and $\Phi(\nu)$ need be considered.

Just as with Fourier series, the function $\Phi(\nu)$ can be recovered from $F(t)$ by inversion. This is the cornerstone of Fourier theory because, astonishingly, the inversion has exactly the same form as the synthesis, and we can write, if $\Phi(\nu)$ is real and $F(t)$ is symmetrical,

$$\Phi(\nu) = \int_{-\infty}^{\infty} F(t)\cos(2\pi\nu t)dt, \tag{1.14}$$

so that not only is $\Phi(\nu)$ the Fourier transform of $F(t)$, but also $F(t)$ is the Fourier transform of $\Phi(\nu)$. The two together are called a 'Fourier pair'.

The complete and rigorous proof of this is long and tedious[3] and it is not necessary here; but the formal definition can be given and this is a suitable place to abandon, for the moment, the physical variables time and frequency and to change to the pair of abstract variables, x and p, which are usually used. The formal statement of a Fourier transform is then

$$\Phi(p) = \int_{-\infty}^{\infty} F(x)e^{2\pi ipx}\,dx, \tag{1.15}$$

$$F(x) = \int_{-\infty}^{\infty} \Phi(p)e^{-2\pi ipx}\,dp \tag{1.16}$$

and this pair of formulae[4] will be used from here on.

[3] It is to be found, for example, in E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford, 1962 or in R. R. Goldberg, *Fourier Transforms*, Cambridge University Press, Cambridge, 1965.

[4] Sometimes one finds

$$\Phi(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)e^{ipx}\,dx; \qquad F(x) = \int_{-\infty}^{\infty} \Phi(p)e^{-ipx}\,dp$$

as the defining equations, and again symmetry is preserved by some people by defining the transform by

$$\Phi(p) = \left\{\frac{1}{2\pi}\right\}^{1/2}\int_{-\infty}^{\infty} F(x)e^{ipx}\,dx; \qquad F(x) = \left\{\frac{1}{2\pi}\right\}^{1/2}\int_{-\infty}^{\infty} \Phi(p)e^{-ipx}\,dp.$$

Symbolically we write

$$\Phi(p) \rightleftharpoons F(x).$$

One and only one of the integrals must have a minus sign in the exponent. Which of the two you choose does not matter, so long as you keep to the rule. If the rule is broken half way through a long calculation the result is chaos; but if someone else has used the opposite choice, the Fourier pair calculated of a given function will be the complex conjugate of that given by your choice.

When time and frequency are the conjugate variables we shall use

$$\Phi(\nu) = \int_{-\infty}^{\infty} F(t)e^{-2\pi i\nu t}\, dt, \tag{1.17}$$

$$F(t) = \int_{-\infty}^{\infty} \Phi(\nu)^{2\pi i\nu t}\, d\nu \tag{1.18}$$

and again, symbolically,

$$\Phi(\nu) \rightleftharpoons F(t).$$

There are two good reasons for incorporating the 2π into the exponent. Firstly the defining equations are easily remembered without worrying where the 2π's go, but, more importantly, quantities like t and ν are actually physically measured quantities – time and frequency – rather than time and *angular* frequency, ω. Angular measure is for mathematicians. For example, when one has to integrate a function wrapped around a cylinder it is convenient to use the angle as the independent variable. Physicists will generally find it more convenient to use t and ν, for example, with the 2π in the exponent.

1.5 Conjugate variables

Traditionally x and p are used when abstract transforms are considered and they are called 'conjugate variables'. Different fields of physics and engineering use different pairs, such as frequency, ν, and time, t, in acoustics, telecommunications and radio; position, x, and momentum divided by Planck's constant, $p/\hbar$, in quantum mechanics; and aperture, x, and the sine of the diffraction angle divided by the wavelength, $p = \sin\theta/\lambda$, in diffraction theory.

In general we will use x and p as abstract entities and give them a physical meaning when an illustration seems called for. It is worth remembering that x and p have inverse dimensionality, as in time, t, and frequency, t^{-1}. The product px, like any exponent, is always a dimensionless number.

One further definition is needed: the 'power spectrum' of a function.[5] This notion is important in electrical engineering as well as in physics. If power

[5] Actually the *energy* spectrum; 'power spectrum' is just the conventional term used in most books. This is discussed in more detail in Chapter 4.

is transmitted by electromagnetic radiation (radio waves or light) or by wires or waveguides, the voltage at a point varies with time as $V(t)$. $\Phi(\nu)$, the Fourier transform of $V(t)$, may very well be – indeed usually is – complex. However, the power per unit frequency interval being transmitted is proportional to $\Phi(\nu)\Phi^*(\nu)$, where the constant of proportionality depends on the load impedance. The function $S(\nu) = \Phi(\nu)\Phi^*(\nu) = |\Phi(\nu)|^2$ is called the power spectrum or the spectral power density (SPD) of $F(t)$. This is what an optical spectrometer measures, for example.

1.6 Graphical representations

It frequently happens that greater insight into the physical processes which are described by a Fourier transform can be achieved by use of a diagram rather than a formula. When a real function $F(x)$ is transformed it generally produces a complex function $\Phi(p)$, which needs an Argand diagram to demonstrate it. Three dimensions are required: Re $\Phi(p)$, Im $\Phi(p)$ and p. A perspective drawing will display the function, which appears as a more or less sinuous line. If $F(x)$ is symmetrical, the line lies in the Re p-plane, whereas if it is antisymmetrical, the line lies in the Im p-plane. Figures 8.1 and 8.2 in Chapter 8 illustrate this point.

Electrical engineering students, in particular, will recognize the end-on view along the p-axis as the 'Nyquist diagram' of feedback theory. There will be examples of this graphical representation in later chapters.

1.7 Useful functions

There are some functions which occur again and again in physics, and whose properties should be learned. They are extremely useful in the manipulation and general taming of other functions which would otherwise be almost unmanageable. Chief among these are the following.

1.7.1 The 'top-hat' function[6]

This has the property that

$$\Pi_a(x) = \begin{cases} 0, & -\infty < x < -a/2 \\ 1, & -a/2 < x < a/2 \\ 0, & a/2 < x < \infty \end{cases}$$

and the symbol Π is chosen as an obvious aid to memory.

[6] In the USA this is called a 'box-car' or 'rect' function.

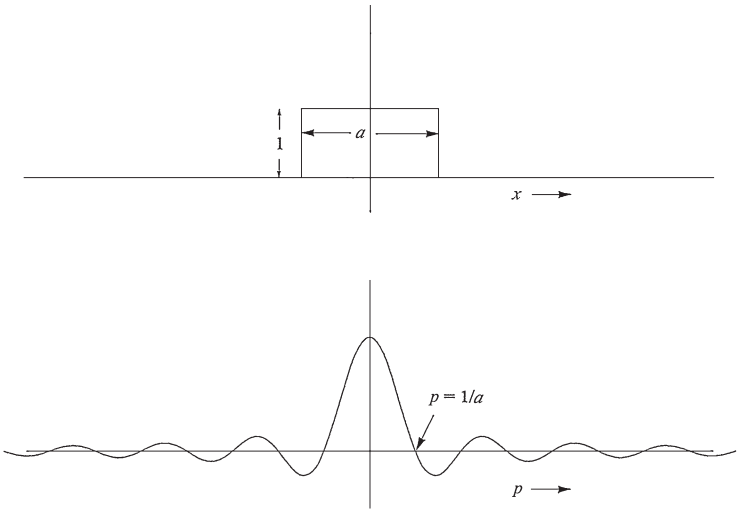

Fig. 1.5. The top-hat function and its transform, the sinc-function.

Its Fourier pair is obtained by integration:

$$\begin{aligned}\Phi(p) &= \int_{-\infty}^{\infty} \Pi_a(x) e^{2\pi i p x}\, dx \\ &= \int_{-a/2}^{a/2} e^{2\pi i p x}\, dx \\ &= \frac{1}{2\pi i p}[e^{\pi i p a} - e^{-\pi i p a}] \\ &= a\left\{\frac{\sin(\pi p a)}{\pi p a}\right\} \\ &= a\,\mathrm{sinc}(\pi p a)\end{aligned}$$

and the 'sinc-function', defined[7] by $\mathrm{sinc}(x) = \sin x/x$, is one which recurs throughout physics (Fig. 1.5). As before, we write symbolically

$$\Pi_a(x) \rightleftharpoons a\,\mathrm{sinc}(\pi p a).$$

1.7.2 The sinc-function

The sinc-function $\mathrm{sinc}(x) = \sin x/x$ has the value unity at $x = 0$, and has zeros whenever $x = n\pi$. The function $\mathrm{sinc}(\pi p a)$ above, the most common form, has zeros when $p = 1/a, 2/a, 3/a, \ldots$

[7] Caution: some people define $\mathrm{sinc}(x)$ as $\sin(\pi x)/(\pi x)$, although without noticeable advantage and with occasional confusion when the argument is complicated.

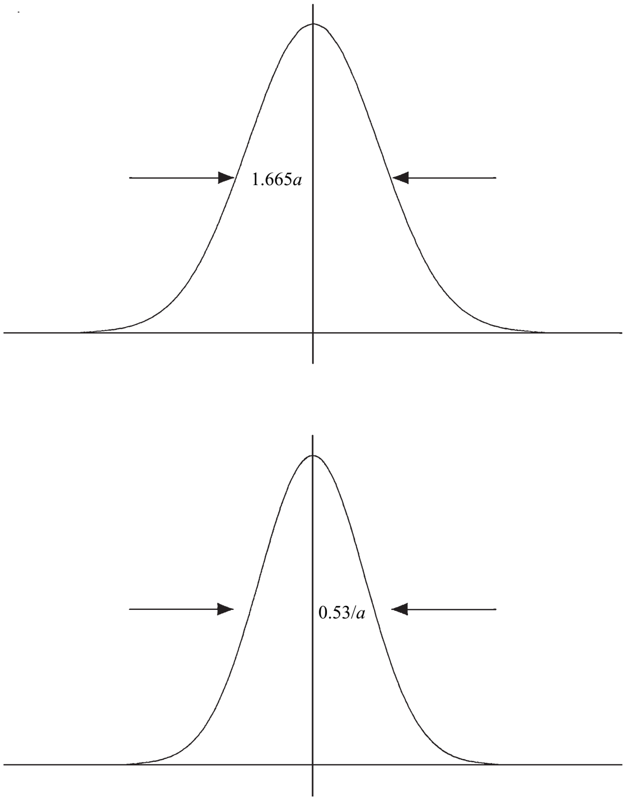

Fig. 1.6. The Gaussian function and its transform, another Gaussian with full width at half maximum inversely proportional to that of its Fourier pair.

1.7.3 The Gaussian function

Suppose $G(x) = e^{-x^2/a^2}$, where a is the 'width parameter' of the function. The value of $G(x) = 1/2$ when $(x/a)^2 = \log_e 2$, or $x = \pm 0.8325a$ so that the full width at half maximum (FWHM) is $1.665a$ and (which every scientist should know!) $\int_{-\infty}^{\infty} e^{-x^2/a^2}\,dx = a\sqrt{\pi}$.

Its Fourier transform is $g(p)$, given by

$$g(p) = \int_{-\infty}^{\infty} e^{-x^2/a^2} e^{2\pi ipx}\,dx$$

(Fig. 1.6). The exponent can be rewritten (by 'completing the square') as

$$-(x/a - \pi ipa)^2 - \pi^2 p^2 a^2$$

and then

$$g(p) = e^{-\pi^2 p^2 a^2} \int_{-\infty}^{\infty} e^{-(x/a - \pi i p a)^2}\, dx.$$

Put $x/a - \pi i p a = z$, so that $dx = a\, dz$. Then

$$\begin{aligned} g(p) &= a e^{-\pi^2 p^2 a^2} \int_{-\infty}^{\infty} e^{-z^2}\, dz \\ &= a\sqrt{\pi} e^{-\pi^2 a^2 p^2} \end{aligned}$$

so that $g(p)$ is another Gaussian function, with width parameter $1/(\pi a)$.

Notice that the wider the original Gaussian, the narrower will be its Fourier pair.

Notice, too, that the value at $p = 0$ of the Fourier pair is equal to the area under the original Gaussian.

1.7.4 The exponential decay

This, in physics, is generally the positive part of the function $e^{-x/a}$. It is asymmetrical, so its Fourier transform is complex:

$$\begin{aligned} \Phi(p) &= \int_{0}^{\infty} e^{-x/a} e^{2\pi i p x}\, dx \\ &= \left[\frac{e^{2\pi i p x - x/a}}{2\pi i p - 1/a} \right]_0^{\infty} = \frac{-1}{2\pi i p - 1/a}. \end{aligned}$$

Usually, with this function, the power spectrum is the most interesting:

$$|\Phi(p)|^2 = \frac{a^2}{4\pi^2 p^2 a^2 + 1}.$$

This is a bell-shaped curve, similar in appearance to a Gaussian curve, and is generally known as a Lorentz profile.[8] Its FWHM is $1/(\pi a)$.

It is the shape found in spectrum lines when they are observed at very low pressure, when collisions between emitting particles are infrequent compared with the transition probability. If the line profile is taken as a function of frequency, $I(\nu)$, the FWHM, $\Delta\nu$, is related to the 'lifetime of the excited state', the reciprocal of the transition probability in the atom which undergoes the transition. In this example, a and x obviously have dimensions of time. Looked

[8] It is also known to mathematicians as the 'Witch of Agnesi' or more accurately as the 'curve of Agnesi', having been studied by the eighteenth-century mathematician Maria Agnesi (1718–1799). The translator confused *versiera* – 'curve' – with *avversiera* – witch.

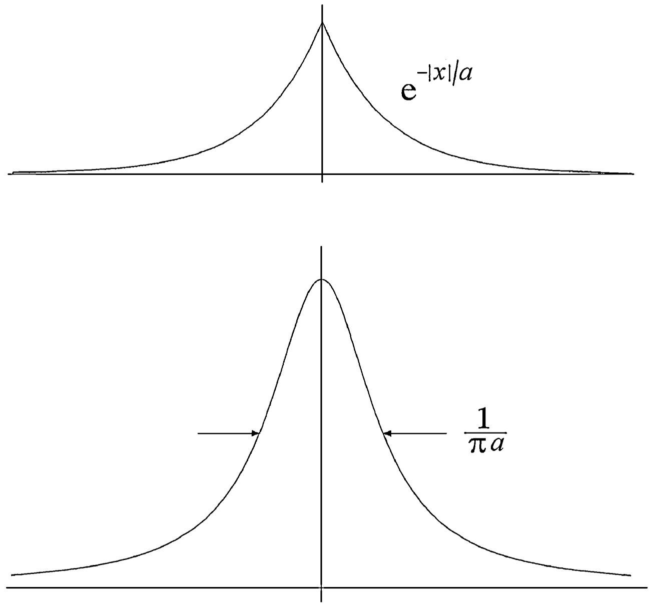

Fig. 1.7. The exponential decay $e^{-|x|/a}$ and its Fourier transform.

at classically, the emitting particle is behaving like a damped harmonic oscillator radiating power at an exponentially decreasing rate. Quantum mechanics yields the same equation through perturbation theory.

There is more discussion of this profile in Chapter 5.

1.7.5 The Dirac 'delta-function'

This has the following properties:

$$\delta(x) = 0 \text{ unless } x = 0,$$

$$\delta(0) = \infty,$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1.$$

It is an example of a function which disobeys one of Dirichlet's conditions, since it is unbounded at $x = 0$. It can be regarded crudely as the limiting case of a top-hat function $(1/a)\Pi_a(x)$ as $a \to 0$. It becomes narrower and higher, and its area, which we shall refer to as its *amplitude*, is always equal to unity. Its Fourier transform (Fig. 1.7) is $\text{sinc}(\pi pa)$ and, as $a \to 0$, $\text{sinc}(\pi pa)$ stretches and in the limit is a straight line at unit height above the x-axis. In other words,

the Fourier transform of a delta-function is unity

and we write

$$\delta(x) \rightleftharpoons 1.$$

Alternatively, and more accurately, it is the limiting case of a Gaussian function of unit area as it gets narrower and higher. Its Fourier transform then is another Gaussian of unit height, getting broader and broader until in the limit it is a straight line at unit height above the axis.

Although the function has infinite height, we frequently encounter it multiplied by a constant. In this case it is convenient, if not strictly accurate, to refer to the function $a\delta(x)$ as having a 'height' a.

The following useful properties of the delta-function (or δ-function) should be memorized. They are

$$\delta(x-a) = 0 \text{ unless } x = a$$

and the so-called 'shift theorem':

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a),$$

where the product under the integral sign is zero except at $x = a$, where, on integration, the δ-function has the amplitude $f(a)$.

It is then easy to show, using this shift theorem, that for positive[9] values of a, b, c and d

$$\delta(x/a - 1) = a\delta(x-a).$$

To show this, put $x = au; dx = a\,du$. Then

$$\int_{-\infty}^{\infty} \delta(x/a-1)f(x)dx = a\int_{-\infty}^{\infty} \delta(u-1)f(au)du$$

and the integrand is zero except at the point $u = 1$, so that the result is $af(a)$. Compare this with

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a)$$

and the substitution is obvious.

[9] For negative values of these quantities a minus sign may be needed, bearing in mind that the integral of a δ-function is always positive, even though a, for example, may be negative. Alternatively, we may write, for example, $\delta(x/a-1) = |a|\delta(x-a)$.

Similarly, we find

$$\delta(a/b - c/d) = ac\delta(ad - bc)$$
$$= bd\delta(ad - bc)$$
$$\delta(ax) = (1/a)\delta(x).$$

Another important consequence of the shift theorem is that

$$\int_{-\infty}^{\infty} e^{2\pi ipx}\delta(x-a)dx = e^{2\pi ipa}$$

so that we can write

$$\delta(x-a) \rightleftharpoons e^{2\pi ipa},$$
$$\delta(mx-a) \rightleftharpoons (1/m)e^{2\pi ipa/m}$$

and a formula which we shall need in Chapter 7:

$$\frac{1}{n}\delta\left(\frac{p}{l}-\frac{r}{n}\right) = \delta\left(\frac{pn}{l}-r\right) \rightleftharpoons e^{-2\pi i(\frac{pn}{l}-r)}.$$

1.7.6 A pair of δ-functions

If two δ-functions are equally disposed on either side of the origin, the Fourier transform is a cosine wave:

$$\delta(x-a)+\delta(x+a) \rightleftharpoons e^{2\pi ipa}+e^{-2\pi ipa}$$
$$= 2\cos(2\pi pa).$$

1.7.7 The Dirac comb

This is an infinite set of equally-spaced δ-functions, usually denoted by the Cyrillic letter Ш (shah). Formally, we write

$$Ш_a(x) = \sum_{n=-\infty}^{\infty} \delta(x-na).$$

It is useful because it allows us to include Fourier series in the general theory of Fourier transforms. For example, the *convolution* (to be described later) of $Ш_a(x)$ and $(1/b)\Pi_b(x)$ (where $b < a$) is a square wave similar to that in the earlier example, of period a and width b, and with unit area in each rectangle. The Fourier transform is then a Dirac comb, with 'teeth' of height a_m spaced at intervals $1/a$. The a_m are, of course, the coefficients in the series.

If the square wave is allowed to become infinitesimally wide and infinitely high so that the area under each rectangle remains unity, then the coefficients a_m

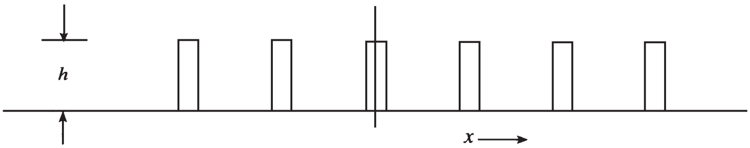

Fig. 1.8. A rectangular pulse-train with a 4 : 1 ‘mark–space’ ratio.

will all become of the same height, $1/a$. In other words, the Fourier transform of a Dirac comb is another Dirac comb:

$$Ш_a(x) \rightleftharpoons \frac{1}{a} Ш_{1/a}(p)$$

and again notice that the period in p-space is the reciprocal of the period in x-space.

This is not a formal demonstration of the Fourier transform of a Dirac comb. A rigorous proof is much more elaborate, but is unnecessary here.

1.8 Worked examples

(1) A train of rectangular pulses, as in Fig. 1.8, has a pulse width equal to 1/4 of the pulse period. Show that the 4th, 8th, 12th etc. harmonics are missing.

Taking zero at the centre of one pulse, the function is clearly symmetrical so that there are only cosine amplitudes:

$$\begin{aligned} A_n &= \frac{2}{P}\int_{-P/8}^{P/8} h\cos\left(\frac{2\pi n x}{P}\right)dx \\ &= \left(\frac{h}{\pi n}\right) 2\sin\left(\frac{2\pi n}{P}\cdot\frac{P}{8}\right) \\ &= \left(\frac{h}{2}\right)\operatorname{sinc}\left(\frac{\pi n}{4}\right) \end{aligned}$$

so that $A_n = 0$ if $n = 4, 8, 12, \ldots$

(2) Find the sine-amplitude of a sawtooth waveform as in Fig. 1.9.

By choosing the origin half way up one of the teeth, the function is clearly made antisymmetrical, so that there are no cosine amplitudes:

$$\begin{aligned} B_n &= \frac{2}{P}\int_{-P/2}^{P/2} \frac{xh}{P}\sin\left(\frac{2\pi n x}{P}\right)dx \\ &= \frac{2h}{P^2}\left[-x\cos\left(\frac{2\pi n x}{P}\right)\frac{P}{2\pi n} + \frac{P^2}{4\pi^2 n^2}\sin\left(\frac{2\pi n x}{P}\right)\right]_{-P/2}^{P/2} \\ &= [-2h/(\pi n)]\cos(\pi n) \end{aligned}$$

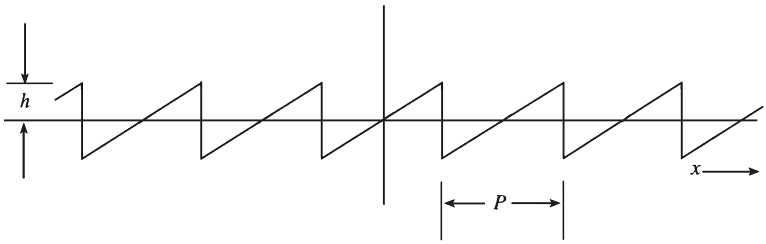

Fig. 1.9. A sawtooth waveform, antisymmetrical about the origin.

since $\sin(\pi n) = 0$, so that

$$B_0 = 0,$$
$$B_n = (-1)^{n+1}[2h/(\pi n)], \quad n \neq 0.$$

As a matter of interest, it is worthwhile calculating the sine-amplitudes when the origin is taken at the tip of a tooth, to see how changing the position of the origin changes the amplitudes. It is also worthwhile doing the calculation for a similar wave, with negative-going slopes instead of positive.