University Maths Challenge 2019/2020
Shuttle questions

Round 1: Questions

1. We throw two fair dice continuously and each time we record their sum. Let P denote the probability that the sum of five will make an appearance before the sum of seven. Find P and pass it on.

2. P can be written as $\frac{A}{B}$, the ratio of two primes. An urn contains 1000 tickets numbered from 1 to 1000. We choose a ticket at random. Let Q denote the probability that the chosen ticket has a number which is a multiple of two or B. Find Q and pass it on.

3. In a box there are $\frac{20}{3} \times Q$ big red balls, 6 small red balls, 6 big yellow balls and N small yellow balls. Compute N so that the size and colour of a ball are independent (for a ball selected at random) and pass it on.

4. Two equally skilled archers alternatively shoot at a target. The first one who hits the target obtains a prize. If the probability of hitting a target with a single shot is $\frac{1}{N-6}$, what is the probability that the archers obtain the prize in the second shoot?

Round 1: Answers

1. Let $E = \{$sum of five appears before the sun of seven$\}$, $A_5 = \{$the first throw results in a sum of five$\}$, $A_7 = \{$the first throw results in a sum of seven$\}$ and $A_{\not 5, \not 7} = \{$the first throw results in a sum which is neither five nor seven$\}$ and consider conditioning on the outcome of the first throw. By the law of total probability, we get

$$\begin{aligned} P = \mathbb{P}(E) &= \mathbb{P}(E \mid A_5)\mathbb{P}(A_5) + \mathbb{P}(E \mid A_7)\mathbb{P}(A_7) + \mathbb{P}(E \mid A_{\not 5, \not 7})\mathbb{P}(A_{\not 5, \not 7}) \\ &= 1 \times \frac{4}{36} + 0 \times \frac{6}{36} + \mathbb{P}(E \mid A_{\not 5, \not 7})\frac{26}{36} \\ &= 1 \times \frac{4}{36} + 0 \times \frac{6}{36} + \mathbb{P}(E)\frac{26}{36}. \end{aligned}$$

   Rearranging, we get $\frac{10}{36}\mathbb{P}(E) = \frac{4}{36}$. Thus $P = \mathbb{P}(E) = \frac{4}{10} = \frac{2}{5}$.

2. Let $A_2 = \{$number of selected ticket is a multiple of two$\}$ and $A_5 = \{$number of selected ticket is a multiple of five$\}$. We are asked to find

$$Q = \mathbb{P}(A_2 \cup A_5) = \mathbb{P}(A_2) + \mathbb{P}(A_5) - \mathbb{P}(A_2 \cap A_5). \tag{1}$$

   We have that $|A_2| = 500$ and $|A_5| = 200$. Thus $\mathbb{P}(A_2) = \frac{500}{1000} = \frac{1}{2}$ and $\mathbb{P}(A_5) = \frac{200}{1000} = \frac{1}{5}$. Also $A_2 \cap A_5 = \{$number of selected ticket is a multiple of two and five$\}$=$\{$number of selected ticket is a multiple of ten$\}$. Thus $|A_2 \cap A_5| = 100$ and $\mathbb{P}(A_2 \cap A_5) = \frac{100}{1000} = \frac{1}{10}$. Substituting in (1), we get that $Q = \frac{1}{2} + \frac{1}{5} - \frac{1}{10} = \frac{6}{10} = \frac{3}{5}$.

3. Let N denote the number of small yellow balls. Then

$$\mathbb{P}(\text{a ball is small}) = \frac{6+N}{16+N},$$
$$\mathbb{P}(\text{a ball is yellow}) = \frac{6+N}{16+N} \quad \text{and}$$
$$\mathbb{P}(\text{a ball is small and yellow}) = \frac{N}{16+N}.$$

For the size and colour of a ball to be independent we need

$$\mathbb{P}(\text{a ball is small and yellow}) = \mathbb{P}(\text{a ball is small}) \times \mathbb{P}(\text{a ball is yellow}).$$

Thus

$$\frac{N}{16+N} = \frac{6+N}{16+N} \times \frac{6+N}{16+N} \Rightarrow (16+N)N = (6+N)^2$$
$$\Rightarrow 16N + N^2 = N^2 + 12N + 36$$
$$\Rightarrow 4N = 36$$
$$\Rightarrow N = 9.$$

4. Let E ={archers obtain the prize in the second shoot}. Then $E = E_1 \cup E_2$ where E_1 ={archer 1 obtains the prize in the second shoot} and E_2 ={archer 2 obtains the prize in the second shoot} where E_1 and E_2 are mutually exclusive. Then

$$\mathbb{P}(E) = \mathbb{P}(E_1) + \mathbb{P}(E_2) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{9} \times (\frac{1}{3} + \frac{2}{9}) = \frac{20}{81}.$$

Round 2: Questions

1. Twelve boys and nine girls are in a room. They are leaving the room one by one. What is the probability that the last person to leave the room is a girl? Find this probability and express it as $\frac{p}{q}$, where p and q are primes. Pass on p and q.

2. Using sampling with replacement we choose five digits from the set $\{0, 1, \ldots, 9\}$. Let P denote the probability that digit d, where $d = q \pmod p$, is the greatest digit chosen. P can be expressed as $\frac{a}{b}$ where a is a prime. Find a and pass it on.

3. Ava and Bob are playing the following game: Ava rolls a fair die once and Bob has to guess what number she rolled. Bob keeps guessing until he guesses Ava's number. If he guesses in the first attempt he gets £$(a - 26)$, if he guesses in the second attempt he gets £4, if he guesses in the third attempt he gets £3, if he guesses in the fourth attempt he gets £2, if he guesses in the fifth attempt he gets £1 and if he guesses in the sixth attempt he gets £0. Let E denote Bob's expected winnings. Find E and pass it on.

4. The probability that it rains tomorrow is $\frac{1}{E}$. The probability that it rains the day after tomorrow is $\frac{2}{E}$. Let R denote the event that it rains either tomorrow or the day after tomorrow. What is the minimum value that $\mathbb{P}(R)$ can take?

Round 2: Answers

1. The corresponding probability is $P = \frac{9}{21} = \frac{3}{7}$.

2. Let d_i denote the i^{th} digit chosen ($i = 1, \ldots, 5$). Then

$$\begin{aligned} P &= \mathbb{P}(\max\{d_1, d_2, d_3, d_4, d_5\} = 1) \\ &= \mathbb{P}(\max\{d_1, d_2, d_3, d_4, d_5\} \le 1) - \mathbb{P}(\max\{d_1, d_2, d_3, d_4, d_5\} \le 0) \\ &= \mathbb{P}(d_i \le 1)^5 - \mathbb{P}(d_i \le 0)^5 \\ &= \left(\frac{2}{10}\right)^5 - \left(\frac{1}{10}\right)^5 \\ &= \frac{2^5 - 1}{10^5} \\ &= \frac{31}{10^5}. \end{aligned}$$

3.

$$\mathbb{P}(\text{Bob guesses in the first attempt}) = \frac{1}{6}.$$

$$\mathbb{P}(\text{Bob guesses in the second attempt}) = \frac{5}{6} \times \frac{1}{5} = \frac{1}{6}.$$

$$\mathbb{P}(\text{Bob guesses in the third attempt}) = \frac{5}{6} \times \frac{4}{5} \times \frac{1}{4} = \frac{1}{6}.$$

$$\mathbb{P}(\text{Bob guesses in the fourth attempt}) = \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{6}.$$

$$\mathbb{P}(\text{Bob guesses in the fifth attempt}) = \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{6}.$$

$$\mathbb{P}(\text{Bob guesses in the sixth attempt}) = \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1 = \frac{1}{6}.$$

Hence,

$$E = \mathbb{E}(\text{winning}) = \sum_{k=1}^{6}(6-k)\mathbb{P}(\text{Bob guesses in the } k^{th} \text{ attempt})$$

$$= \sum_{k=1}^{6}(6-k)\frac{1}{6} = \frac{1}{6}(1+2+3+4+5) = \frac{15}{6} = \frac{5}{2}.$$

4. Let $A = \{\text{it rains tomorrow}\}$ and $B = \{\text{it rains the day after tomorrow}\}$. Then

$$\mathbb{P}(R) = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

$\mathbb{P}(A \cap B)$ ranges from 0 (when A and B are disjoint) to $\frac{1}{E}$ (when $A \subset B$). Thus $\mathbb{P}(R)$ takes the minimum value when $\mathbb{P}(A \cap B) = \frac{1}{E}$. Thus the minimum value is calculated as

$$\mathbb{P}(R) = \frac{1}{E} + \frac{2}{E} - \frac{1}{E} = \frac{2}{E} = \frac{4}{5}.$$