University Maths Challenge 2019/2020
Round 1

Question 1

There are three green balls, three black balls and three purple balls in a box. Two balls are drawn out from the box at random. If one of the balls taken out from the box is green, what is the probability that the other ball is also green?

Question 2

A fair die is rolled ten times. Compute the probability of getting exactly two 2s, three 3s and two 4s.

Question 3

Four children stand in a circle, each having a ball. All of the children simultaneously throw their ball to another of the children, with each child choosing at random who to throw to. What is the probability that each child has exactly one ball thrown to them?

Question 4

The probability that a T-shirt is in a chest of drawers is p. If it is in the chest of drawers, then it has equal probability to be in any one of the five drawers of the chest of drawers. We looked in the four drawers and did not find it. What is the probability that the T-shirt is in the fifth drawer?

Question 5

n balls are distributed at random into r boxes ($n, r \in \mathbb{N}$). Compute the probability that there are exactly k balls in the first j boxes ($1 \leq k \leq n$, $1 \leq j \leq r$).

Tie-break question

n balls are distributed at random into r boxes ($n, r \in \mathbb{N}$). Suppose $n \leq r$. Compute the probability that no box contains more than one ball.

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Round 1 Solutions

Question 1

$$\frac{1}{7}$$

Question 2

$$\binom{10}{2}\binom{8}{3}\binom{5}{2}\left(\frac{1}{6}\right)^7\left(\frac{3}{6}\right)^3 = \frac{10!}{2!2!3!3!}\frac{3^3}{6^{10}}$$

Question 3

$$\frac{9}{3^4} = \frac{1}{81}$$

Question 4

$$\frac{p}{5-4p}$$

Question 5

$$\binom{n}{k}\ \left(\frac{j}{r}\right)^k\ \left(1-\frac{j}{r}\right)^{n-k}$$

Tie break question

$$\frac{\binom{r}{n}n!}{r^n}$$

University Maths Challenge 2019/2020
Round 1
Worked out solutions

Question 1

Consider the events

$$A = \{\text{at least one ball is green}\}$$
$$B = \{\text{both balls are green}\}$$

We want to compute

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B)}{\mathbb{P}(A)}.$$

We have that

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{\binom{6}{2}}{\binom{9}{2}} = 1 - \frac{15}{36} = \frac{21}{36} = \frac{7}{12}$$

and that

$$\mathbb{P}(B) = \frac{\binom{3}{2}}{\binom{9}{2}} = \frac{3}{36} = \frac{1}{12}.$$

Hence,

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B)}{\mathbb{P}(A)} = \frac{\frac{1}{12}}{\frac{7}{12}} = \frac{1}{7}.$$

Question 2

$$\binom{10}{2}\binom{8}{3}\binom{5}{2}\left(\frac{1}{6}\right)^7\left(\frac{3}{6}\right)^3$$

Question 3

Number the children $1, 2, 3, 4$. Each child chooses randomly who to throw the ball to out of the three other children, so each child has three possible choices. Let Ω be the sample space of the experiment. Then

$$\Omega = \Big\{(w_1, w_2, w_3, w_4) : w_1 \in \{2, 3, 4\}, w_2 \in \{1, 3, 4\}, w_3 \in \{1, 2, 4\}, w_4 \in \{1, 2, 3\}\Big\},$$

where w_k denotes the number of the child that child k threw the ball to, $k = 1, 2, 3, 4$. Thus, the number of all possible outcomes is $|\Omega| = 3^4$. The event that each child has exactly one ball thrown to them is the set of outcomes which are permutations of the set $1, 2, 3, 4$ with no matches: we say that a match occurs in a permutation of $1, 2, 3, 4$ if number k is at the k^{th} position, $k = 1, 2, 3, 4$.

For example, the permutation $(1, 3, 4, 2)$ has a match because 1 is in the 1^{st} position. Permutation $(2, 1, 3, 4)$ has two matches. However, permutations $(2, 3, 4, 1)$ and $(3, 1, 4, 2)$ have no matches. In total there are 9 permutation of $\{1, 2, 3, 4\}$ with no matches:

$$\begin{array}{ccc} (2,1,4,3) & (3,1,4,2) & (4,3,2,1) \\ (2,3,4,1) & (3,4,1,2) & (4,3,1,2) \\ (2,4,1,3) & (3,1,2,4) & (4,1,2,3) \end{array}$$

Thus, the probability that each child has exactly one ball thrown to them is equal to

$$\frac{9}{3^4} = \frac{1}{81}.$$

Question 4

Consider the events

$$A = \{\text{the T-shirt is not in the four drawers}\}$$
$$B = \{\text{the T-shirt is in the fifth drawer}\}$$

We want to compute

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

We have that

$$\mathbb{P}(A) = (1-p) + p \times \frac{1}{5}.$$

In addition,

$$\mathbb{P}(A \cap B) = \mathbb{P}(B) = p \times \frac{1}{5}.$$

Hence,

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{p \times \frac{1}{5}}{(1-p) + p \times \frac{1}{5}} = \frac{p}{5-4p}.$$

Question 5

Number the balls from 1 to n. Each ball chooses at random into which box it goes. The probability that a ball falls into any of the first j boxes is equal to $\frac{j}{r}$ since the boxes are equally probable to be chosen. If we want k balls to be in the first j boxes than first we first choose which k balls go into the first j boxes. There are $\binom{n}{k}$ ways to choose k balls out of n to go into the first j boxes. The probability that those k balls are in the first j boxes is equal to $\left(\frac{j}{r}\right)^k$ because balls choose which box to go to independently of each other. Finally, the probability that the remaining $n-k$ balls are not in the first j boxes is equal to $\left(1-\frac{j}{r}\right)^{n-k}$. Therefore, the probability that there are exactly k balls in the first j boxes is equal to

$$\binom{n}{k} \left(\frac{j}{r}\right)^k \left(1-\frac{j}{r}\right)^{n-k}$$

Tie-break question

$$\frac{\binom{r}{n} n!}{r^n}$$