CE-370 Safety and Reliability of Engineering Systems

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SUMMARY OF LECTURE 11 - FUNCTIONS OF MULTIPLE RANDOM VARIABLES

Joint CDF and PDF of Multiple Random Variables

The cumulative distribution function of n random variables $X_1, \ldots, X_n$ is defined as

$$F_{X_1 \ldots X_n}(x_1, \ldots, x_n) = P[X_1 \leq x_1, \ldots, X_n \leq x_n] \tag{1}$$

where the comma should be interpreted as an "and" statement, i.e. $P[A, B] = P[A \cap B]$. By definition, the joint probability density function is defined as

$$f_{X_1 \ldots X_n} = \frac{\partial^n}{\partial x_1 \ldots \partial x_n} F_{X_1 \ldots X_n}(x_1, \ldots, x_n) \tag{2}$$

From this it follows that if $f_{X, \ldots, X_i, \ldots, X_n}(x_1, \ldots, -\infty, \ldots, x_n) = 0$ for any i, then

$$F_{X_1 \ldots X_n}(x_1, \ldots, x_n) = P[X_1 \leq x_1, \ldots, X_n \leq x_n] = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1 \ldots X_n}(\xi_1, \ldots, \xi_n) d\xi_n \ldots d\xi_1 \tag{3}$$

By defining the n-dimensional vector $\mathbf{x} = [x_1 \; x_2 \; \ldots \; x_n]^T$ and a region $\mathcal{R} \subset \mathbb{R}^n$, the probability for any event A defined by $\mathbf{x} \in \mathcal{R}$ can be expressed as

$$P[A] = P[\mathbf{x} \in \mathcal{R}] = \int \cdots \int_{\mathcal{R}} f_{X_1 \ldots X_n}(\xi_1, \ldots, \xi_n) d\xi_n \ldots d\xi_1 \tag{4}$$

Clearly if $\mathcal{R} = \mathbb{R}^n$ then

$$P[X_1 \leq \infty, \ldots, X_n \leq \infty] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1 \ldots X_n}(x_1, \ldots, \xi_n) d\xi_n \ldots d\xi_1 = 1 \tag{5}$$

which is a necessary condition from the basic axioms of probability (see Lecture Notes No.2).

Joint Marginal CDF and PDF of Multiple Random Variables

Conceptually, marginalizing a random variable from a full joint distribution, means determining the joint probability measure of the remaining variables for all possible values of the marginalized variable. Mathematically this can be achieved by integrating out the marginalized variable, such that without loss of generality marginalizing the n^{th} component of a joint distribution we obtain

$$F_{X_1 \ldots X_{n-1}}(x_1, \ldots, x_{n-1}) = P[X_1 \leq x_1, \ldots, X_{n-1} \leq x_{n-1}] = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_{n-1}} \int_{-\infty}^{\infty} f_{X_1 \ldots X_n}(\xi_1, \ldots, \xi_n) d\xi_n \ldots d\xi_1 \tag{6}$$

Consequently

$$f_{X_1 \ldots X_{n-1}}(x_1, \ldots, x_{n-1}) = \int_{-\infty}^{\infty} f_{X_1 \ldots X_n}(x_1, \ldots, \xi_n) d\xi_n \tag{7}$$

By induction one can proceed to marginilize any number of variables from the initial full joint distribution.

Conditional Probability of Multiple Random Variables

If one is interested in computing the conditional probability

$$P[X_1 \leq x_1, \ldots, X_k \leq x_k | x_{k+1}^{(-)} \leq X_{k+1} \leq x_{k+1}^{(+)}, \ldots, x_n^{(-)} \leq X_n \leq x_n^{(+)}] \tag{8}$$

By defining $A = [X_1 \leq x_1, \ldots, X_k \leq x_k]$ and $B = [x_{k+1}^{(-)} \leq X_{k+1} \leq x_{k+1}^{(+)}, \ldots, x_n^{(-)} \leq X_n \leq x_n^{(+)}]$ we can apply Baye's rule

$$P(A|B) = \frac{P(A, B)}{P(B)} \tag{9}$$

to obtain

$$P(A|B) = \frac{P[X_1 \leq x_1, ..., X_k \leq x_k, x_{k+1}^{(-)} \leq X_{k+1} \leq x_{k+1}^{(+)}, ..., x_n^{(-)} \leq X_n \leq x_n^{(+)}]}{P[x_{k+1}^{(-)} \leq X_{k+1} \leq x_{k+1}^{(+)}, ..., x_n^{(-)} \leq X_n \leq x_n^{(+)}]} =$$

$$\frac{\int_{x_n^{(-)}}^{x_n^{(+)}} \cdots \int_{x_{k+1}^{(-)}}^{x_{k+1}^{(+)}}, \int_{-\infty}^{x_k} \cdots \int_{-\infty}^{x_1} f_{X_1 \ldots X_n}(\xi_1, ..., \xi_k, \xi_{k+1}, ..., \xi_n) d\xi_1 ... d\xi_k d\xi_{k+1} ... d\xi_n}{\int_{x_n^{(-)}}^{x_n^{(+)}} \cdots \int_{x_{k+1}^{(-)}}^{x_{k+1}^{(+)}} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f_{X_1 \ldots X_n}(\xi_1, ..., \xi_n) d\xi_1 ... d\xi_k d\xi_{k+1} ... d\xi_n} =$$

$$\frac{\int_{x_n^{(-)}}^{x_n^{(+)}} \cdots \int_{x_{k+1}^{(-)}}^{x_{k+1}^{(+)}}, \int_{-\infty}^{x_k} \cdots \int_{-\infty}^{x_1} f_{X_1 \ldots X_n}(\xi_1, ..., \xi_k, \xi_{k+1}, ..., \xi_n) d\xi_1 ... d\xi_k d\xi_{k+1} ... d\xi_n}{\int_{x_n^{(-)}}^{x_n^{(+)}} \cdots \int_{x_{k+1}^{(-)}}^{x_{k+1}^{(+)}} f_{X_{k+1} \ldots X_n}(\xi_{k+1}, ..., \xi_n) d\xi_{k+1} ... d\xi_n} \tag{10}$$

For the special case of two random variables the previous expression simplifies to

$$P[X_1 \leq x_1 | x_2^{(-)} \leq X_2 \leq x_2^{(+)}] = \frac{P[X_1 \leq x_1, x_2^{(-)} \leq X_2 \leq x_2^{(+)}]}{P[x_2^{(-)} \leq X_2 \leq x_2^{(+)}]} =$$

$$\frac{\int_{x_2^{(-)}}^{x_2^{(+)}} \int_{-\infty}^{x_1} f_{X_1 X_2}(\xi_1 \xi_2) d\xi_1 d\xi_2}{\int_{x_2^{(-)}}^{x_2^{(+)}} \int_{-\infty}^{\infty} f_{X_1 X_2}(\xi_1 \xi_2) d\xi_1 d\xi_2} = \frac{\int_{x_2^{(-)}}^{x_2^{(+)}} \int_{-\infty}^{x_1} f_{X_1 X_2}(\xi_1 \xi_2) d\xi_1 d\xi_2}{\int_{x_2^{(-)}}^{x_2^{(+)}} f_{X_2}(\xi_2) d\xi_2} \tag{11}$$

Conditional CDF and PDF of Multiple Random Variables

From the previous analysis we can define the conditional PDF of k random variables upon knowledge of n-k random variables as

$$f_{X_1...X_k}(x_1,...,x_k|X_{k+1}=x_{k+1},...X_n=x_n)=\frac{f_{X_1...X_n}(x_1,...,x_k,x_{k+1},...,x_n)}{f_{X_{k+1}...X_n}(x_{k+1},...,x_n)} \tag{12}$$

For the special case of two joint random variables the previous expression simplifies to

$$f_{X_1}(x_1|X_2=x_2=a)=\frac{f_{X_1X_2}(x_1,a)}{f_{X_2}(a)} \tag{13}$$

Note that in the right hand side of the previous eq. $x_2=a$ is a fixed number and $f_{X_2}(\cdot)$ is the marginal distribution of X_2.

Independent Random Variables

From the previous analysis and from the axioms of probability it follows that if a set of n random variables is independent then

$$F_{X_1...X_n}(x_1,...x_n)=F_{X_1}(x_1)...F_{X_n}(x_n) \tag{14}$$

and consequently

$$f_{X_1...X_n}(x_1,...x_n)=f_{X_1}(x_1)...f_{X_n}(x_n) \tag{15}$$

Joint CDF and PDF of Functions of Multiple Random Variables

In many situations one is interested in determining the cumulative distribution function (CDF) and probability density function (PDF) of a certain function of a set of random variables with known CDF and PDF. That is, given

$$Z=g(X_1,...X_n) \tag{16}$$

we are interested in

$$F_Z(z)=P[Z\leq z] \tag{17}$$

and its derivative

$$f_Z(z)=\frac{d}{dz}F_Z(z) \tag{18}$$

To begin, we must consider all possible values of the random vector $\mathbf{x}=[x_1,\ldots,x_n]^T$ such that for any given value $Z=z$

$$g(x)\leq z \tag{19}$$

we will denote the union of all the regions where the previous condition prevails as $\boldsymbol{R}_z$. To obtain the desired probability we integrate the joint probability density function of the random vector $\mathbf{X}=[X_1,\ldots,X_n]^T$ over the region $\boldsymbol{R}_z$, that is

$$F_Z(z) = P[Z \leq z] = \int \cdots \int_{\boldsymbol{R}_z} f_{X_1 \ldots X_n}(\xi_1, ..., \xi_n) d\xi_1 ... d\xi_n \tag{20}$$

Expected Value

The expected value of a function $g(\cdot)$ of a random vector $\mathbf{X}$ is given by

$$E[g(\mathbf{X})] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_n)\ f_{X_1 \ldots X_n}(x_1, \ldots, \xi_n) d\xi_1 ... d\xi_n \tag{21}$$

As a special case consider $g(x_1, \ldots, x_n) = x_l^k\ x_m^j$ for $l, m \leq n$ and $k, j \in \mathbb{N}$

$$E[x_l^k\ x_m^j] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_l^k\ x_m^j\ f_{X_1 \ldots X_n}(x_1, \ldots, \xi_n) d\xi_1 ... d\xi_n \tag{22}$$

If $k = 1$ and $j = 1$, this is known as the correlation of X_l and X_m.

Covariance

The covariance or joint central moment between two random variables is defined as

$$COV(X_l\ X_m) = E[(X_l - E[X_l])(X_m - E[X_m])] \tag{23}$$

which after some simple algebraic manipulation can be related to their means and correlation as

$$COV(X_l\ X_m) = E[X_l\ X_m] - E[X_l]E[X_m] \tag{24}$$

If two random variables are independent then their covariance is zero, however zero covariance does not imply the variables are independent.

Special Case: Jointly Gaussian Random Variables

An n-component random vector $\mathbf{X}$ is jointly Gaussian iff its joint probability density function has the form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\,|\mathbf{K}|^{1/2}} e^{-\frac{(\mathbf{x}-\mathbf{m})^T \mathbf{K}^{-1} (\mathbf{x}-\mathbf{m})}{2}} \tag{25}$$

where the (l, m) entry of the covariance matrix $\mathbf{K}$ is defined by the covariance of its components as

$$\mathbf{K}_{l,m} = COV(X_l\ X_m) = E[X_l\ X_m] - E[X_l]E[X_m] \tag{26}$$

and the j^{th} component of the mean vector $\mathbf{m}$ is given by

$$\mathbf{m}_j = E[X_j] \tag{27}$$

The correlation coefficient between X_j and X_k is defined by

$$\rho_{X_j, X_k} = \frac{COV(X_j, X_k)}{\sqrt{COV(X_j, X_j) COV(X_k, X_k)}} \tag{28}$$

For the highly specialized case for only two jointly Gaussians, the joint PDF is given by

$$f_{XY}(x,y) = \frac{\exp\left\{\frac{-1}{2(1-\rho^2)}\left(\left(\frac{x-m_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-m_x}{\sigma_x}\right)\left(\frac{y-m_y}{\sigma_y}\right) + \left(\frac{y-m_y}{\sigma_y}\right)^2\right)\right\}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \tag{29}$$

where $m_x = E[X]$, $m_y = E[Y]$, $\sigma_x^2 = COV(X,X)$ and $\sigma_y^2 = COV(Y,Y)$ and ρ is the correlation coefficient between X and Y. It can be shown that all n marginals of an n dimensional jointly Gaussian random vector are Gaussians with respective mean m_i and variance σ_i^2.

SPECIAL CASES OF FUNCTIONS OF TWO RANDOM VARIABLES

In the following we will discuss some special cases of functions of two random variables

Special Case 1: Sum of Two Random Variables

For the random variable Z resulting from the sum of two random variables X_1 and X_2, the event $Z \leq z$ can found by determining the probability measure of the semi-infinite area defined by $x_1 + x_2 \leq z$. It can then be shown that the CDF and PDF of the random variable $Z = X_1 + X_2$ is given by

$$F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x_1} f_{X_1X_2}(x_1,x_2)dx_2dx_1 \tag{30}$$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f_{X_1X_2}(x_1, z-x_1)dx_1 \tag{31}$$

If X_1 and X_2 are independent, then the above integral simplifies to the following *convolution* integral

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f_{X_1}(x_1)f_{X_2}(z-x_1)dx_1 \tag{32}$$

Special Case 2: Difference of Two Random Variables

In similar fashion to the previous case, the CDF and PDF of the random variable $Z = X_1 - X_2$ can be expressed as

$$F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z+x_2} f_{X_1X_2}(x_1,x_2)dx_1dx_2 \tag{33}$$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f_{X_1X_2}(z+x_2, x_2)dx_2 \tag{34}$$

If X_1 and X_2 are independent, then the above integral simplifies to the following *correlation* integral

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f_{X_2}(x_2)f_{X_1}(z+x_2)dx_2 \tag{35}$$

Special Case 3: Product of Two Random Variables

For a random variable Z resulting from the product of two random variables X_1 and X_2, the event $Z \leq z$ can be found by determining the probability measure of the subset of $\mathbb{R}^2$ defined by $x_1 x_2 \leq z$. It can then be shown that the CDF and PDF of the random variable $Z = X_1 X_2$ is given by

$$F_Z(z) = \int_{-\infty}^{0} dx_1 \int_{z/x_1}^{\infty} f_{X_1X_2}(x_1, x_2)dx_2 + \int_{0}^{\infty} dx_1 \int_{-\infty}^{z/x_1} f_{X_1X_2}(x_1, x_2)dx_2 \tag{36}$$

$$f_Z(z) = \int_{0}^{\infty} \frac{1}{x_1} f_{X_1X_2}(x_1, \frac{z}{x_1})dx_1 - \int_{-\infty}^{0} \frac{1}{x_1} f_{X_1X_2}(x_1, \frac{z}{x_1})dx_1 \tag{37}$$

Note that in the previous result one must be careful to properly account for the sign of x_1 and x_2. In the special case where X_1 and X_2 are independent we obtain

$$f_Z(z) = \int_{0}^{\infty} \frac{1}{x_1} f_{X_1}(x_1) f_{X_2}\left(\frac{z}{x_1}\right) dx_1 - \int_{-\infty}^{0} \frac{1}{x_1} f_{X_1}(x_1) f_{X_2}\left(\frac{z}{x_1}\right) dx_1 \tag{38}$$

Special Case 4: Ratio of Two Random Variables

A similar analysis can be carried out for the random variable Z resulting from the ratio X_1/X_2 of two random variables X_1 and X_2 to obtain

$$f_Z(z) = \int_{0}^{\infty} x_2 f_{X_1,X_2}(zx_2, x_2)dx_2 - \int_{-\infty}^{0} x_2 f_{X_1,X_2}(zx_2, x_2)dx_2 \tag{39}$$

If X_1 and X_2 are independent then

$$f_Z(z) = \int_{0}^{\infty} x_2 f_{X_1}(zx_2) f_{X_2}(x_2)dx_2 - \int_{-\infty}^{0} x_2 f_{X_1}(zx_2) f_{X_2}(x_2)dx_2 \tag{40}$$

Special Case 5: Maximum of Two Random Variables

Let the random variable Z be defined as $Z = max(X_1, X_2)$, then

$$F_Z(z) = F_{X_1X_2}(z, z) \tag{41}$$

This simply means that the event $Z \leq z$ means that both X_1 and X_2 must be less than z. By definition

$$F_Z(z) = F_{X_1X_2}(z, z) = \int_{-\infty}^{z} \int_{-\infty}^{z} f_{X_1X_2}(x_1, x_2)dx_1 \ dx_2 \tag{42}$$

Differentiating with respect to z and using Leibnitz differentiation rule we obtain

$$f_Z(z) = \frac{d}{dz} F_z(z) = \int_{\infty}^{z} f_{X_1X_2}(z, x_2)dx_2 + \int_{\infty}^{z} f_{X_1X_2}(x_1, z)dx_1 \tag{43}$$

If X_1 and X_2 are independent then

$$F_Z(z) = F_{X_1}(z) \ F_{X_2}(z) \tag{44}$$

Moreover if the have the same distribution (i.i.d) then

$$F_Z(z) = (F_X(z))^2 \tag{45}$$

This result can be easily generalized to n i.i.d random variables and thus, the CDF of their maximum is given by the expression

$$F_Z(z) = (F_X(z))^n \tag{46}$$

Special Case 5: Minimum of Two Random Variables

Let the random variable Z be defined as $Z = min(X_1, X_2)$, then

$$F_Z(z) = F_{X_1}(z) + F_{X_2}(z) - F_{X_1 X_2}(z, z) \tag{47}$$

This expression is obtained by using the basic axioms of probability $P(A \cup B) = P(A) + (B) - P(A, B)$, where A is the event $X_1 \leq z$ and B corresponds to the event $X_2 \leq z$. Differentiation yields

$$f_Z(z) = f_{X_1}(z) + f_{X_2}(z) - \frac{d}{dz} F_{X_1 X_2}(z, z) \tag{48}$$

If X_1 and X_2 are independent, the previous expression simplifies to

$$f_Z(z) = f_{X_1}(z)\,(1 - F_{X_2}(z)) + f_{X_2}(z)\,(1 - F_{X_1}(z)) \tag{49}$$