CE-370 Safety and Reliability of Engineering Systems

Instructor: Eric M. Hernandez, PhD

FROM BERNOULLI TO POISSON

Bernoulli Random Variable

The sample space for a Bernoulli random variable X is binary, typically it is defined by $\Omega = [0, 1]$.

$$P[X = 1] = p \tag{1}$$
$$P[X = 0] = 1 - p \tag{2}$$

where $0 < p < 1$. Alternative sample spaces can also be used. Consider for example, $\Omega =$[machine working, machine not working]. Needless to say that all binary sample space are equivalent and thus without loss of generality, we shall retain the traditional $\Omega = [0, 1]$. Typically the outcome $X = 1$ is associated with a success and $X = 0$ with a failure.

Binomial Random Variable

A binomial random variable Y is defined as the number of k successes in n independent realizations of identical Bernoulli random variables X. Alternatively it can also be defined as

$$Y = \sum_{i=1}^{n} X_i \tag{3}$$

The probability of the event $Y = k$ is given by

$$P[Y = k] = \frac{n!}{k!(n-k)!}(1-p)^{n-k}p^k \tag{4}$$

Poisson Random Variable

If we imagine each Bernoulli random variable X occurring within an interval of time Δt such that $n\Delta t = T$, and $p = \lambda \Delta t$, then

$$P[Y = k] = \frac{n!}{k!(n-k)!}\left(1 - \lambda\Delta t\right)^{n-k}\left(\lambda \Delta t\right)^k \tag{5}$$

$$P[Y = k] = \frac{n!}{k!(n-k)!}\left(\frac{n - \lambda T}{n}\right)^{n-k}\left(\frac{\lambda T}{n}\right)^k \tag{6}$$

If we take the limit as $n \to \infty$ and $\Delta t \to 0$ while keeping T fixed, we find

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{n!}{k!(n-k)!}\left(\frac{n-\lambda T}{n}\right)^{n-k}\left(\frac{\lambda T}{n}\right)^{k} && (7)\\
&= \lim_{n\to\infty} \frac{(\lambda T)^k}{k!}\left(1-\frac{\lambda T}{n}\right)^{n} \lim_{n\to\infty} \frac{n!}{(n-k)!\,(n-\lambda T)^k} && (8)\\
&= \frac{(\lambda T)^k\, e^{-\lambda T}}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!\,(n-\lambda T)^k} && (9)\\
P[Y=k] &= \frac{(\lambda T)^k\, e^{-\lambda T}}{k!} && (10)
\end{aligned}
$$

This describes the Poisson random variable where λ describes the probability of success ($X = 1$) per unit of time.

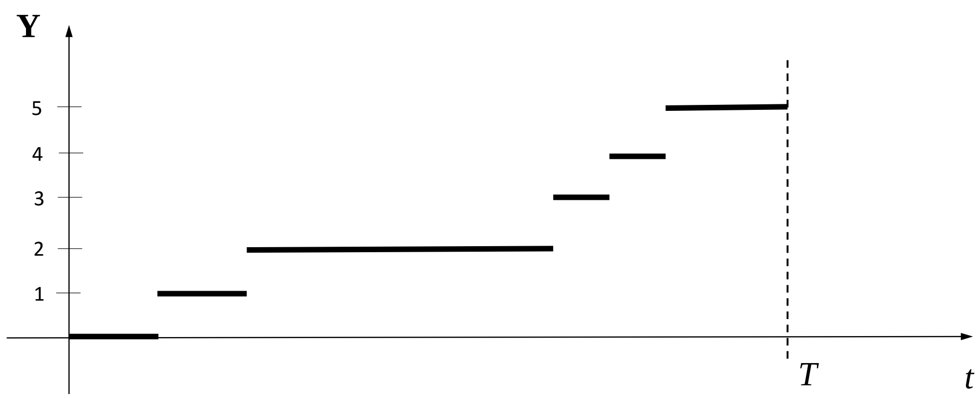

Figure 1: Sample realization of a Poisson random process