CE-370 Safety and Reliability of Engineering Systems

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SUMMARY OF LECTURE 7 - RELIABILITY OF COMPONENTS

Definition

Reliability is defined as the complementary event of Failure. Therefore, if the probability of failure of a given component before or at time t is given by $F(t)$, then based on first principles of probability (see Lecture 2 notes), we find that its reliability is given by

$$R(t) = 1 - F(t) \tag{1}$$

Therefore we can interpret the reliability as the probability of not failing before or at time t, or conversely as the probability of failing at some time after t (assuming failure is assured). Therefore by definition

$$0 \leq R(t) \leq 1 \; \forall t \tag{2}$$

Also note that

$$\frac{dR(t)}{dt} = \frac{d}{dt}(1 - F_T(t)) = -\frac{dF_T(t)}{dt} = -f_T(t) \tag{3}$$

Failure Rate Function

Let's define T as the time of failure and suppose we are interested in the following conditional probability

$$P[t < T \leq t + dt \,|\, T > t] \tag{4}$$

This event corresponds to the probability that a component fails during the infinitesimal interval after t given that it has not failed before. By using the conditional probability rule we find

$$P[t < T \leq t + dt \,|\, T > t] = \frac{f_T(t)dt}{1 - F_T(t)} = \frac{f_T(t)dt}{R(t)} \tag{5}$$

Notice that examined from a population point of view, the previous expression is the ratio between the number of elements that fail between t and t+dt divided by the number of elements that have survived until t.

By defining the *failure rate function* (also known as the hazard function or hazard rate) is given by

$$r(t) = \frac{f_T(t)}{R(t)} \tag{6}$$

The desired conditional probability can be written as

$$P[t < T \leq t + dt \,|\, T > t] = r(t)dt \tag{7}$$

By combining eqs. 3, 5, and the definition of the failure rate function we find that

$$r(t)dt = -\frac{dR}{R(t)} \tag{8}$$

Integrating both sides we obtain an expression for the reliability in terms of the failure rate function

$$R(t) = R(t_o)e^{-\int_{t_o}^{t} r(\tau)d\tau} \tag{9}$$

For the special case where the failure rate is constant, $r(\tau) = \lambda$, $t_o = 0$ and $R(t_o) = 1$, then

$$R(t) = e^{-\int_0^t \lambda d\tau} = e^{-\lambda t} \tag{10}$$

Based on eq. 3, this reliability function corresponds to a failure probability density function equal to

$$f_T(t) = -\frac{dR}{dt} = \lambda e^{-\lambda t} \tag{11}$$

which corresponds to an exponential random variable. Therefore, an exponential random variable is a good model for the probability of failure of components which exhibit a constant failure rate.

Mean Time to Failure

The expected value of T is typically referred to as the mean time to failure (MTTF), and as the name suggests is the time at which a component is expected to fail. The expression for the MTTF is given by

$$E[T] = \int_0^\infty t f_T(t)dt \tag{12}$$

Using integration by parts we find that

$$E[T] = \int_0^\infty t f_T(t)dt = \Big[tF_T(t)\Big]_0^\infty - \int_0^\infty F_T(t)dt = \int_0^\infty (1 - F_T(t))dt \tag{13}$$

$$E[T] = \int_0^\infty R(t)dt \tag{14}$$

Therefore the MTTF is given by the integral of the reliability function. For a component with constant failure rate, the MTTF is given by

$$E[T] = \int_0^\infty e^{-\lambda t}dt = -\frac{1}{\lambda}\left[e^{-\lambda t}\right]_0^\infty = \frac{1}{\lambda} \tag{15}$$

Evaluating the reliability at $t = E[T] = 1/\lambda$ we get

$$R(\frac{1}{\lambda}) = e^{-1} = 0.3679 \tag{16}$$

Therefore it is nearly a two-in-three chance that a component with uniform failure rate will fail before or at the MTTF.