CE-370 Reliability of Engineering Systems

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SUMMARY OF LECTURE 5 - MOMENTS OF A RANDOM VARIABLE

The n^{th} moment of a random variable is defined as

$$\int_{-\infty}^{+\infty} x^n f_X(x)dx$$

where $f_X(x)$ is the probability density function of the random variable X. Two very important moments are the first and second moment, they contain information about the expected value and variance of the random variable.

Expected Value of a Random Variable - First Moment

The expected value $\mu = E[X]$ (also known as the mathematical expectation or mean) is the weighted average of all of the possible values (or infinitesimal intervals) of the random variable.The weight of each value (infinitesimal interval) is given by the probability that the random variable lies on that interval, namelt $f_X(x)dx$. Therefore the expected value $E[X]$ of a random variable is given by

$$\mu_X = \mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)dx$$

As can be seen, by the definition, this coincides with the first moment of the random variable. Graphically the first moment is the centroid of the area delimited by the PDF.

Variance of a Random Variable

The variance ($\sigma_X^2 = Var[X]$) of a random variable measures the spread around the mean of that variable. Typically, the higher the variance, the more difficult it is to predict the random variable. The variance is defined as

$$\sigma_X^2 = \mathbb{E}[(X-\mu)^2] = \int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x)dx$$

Graphically the variance is equivalent to the moment of inertia of the area delimited by the PDF. By expanding the terms inside the expectation, one finds that

$$\sigma_X^2 = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - s\mu\mathbb{E}[X] + \mu^2 = \mathbb{E}[X^2] - \mu^2$$

where

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)dx$$

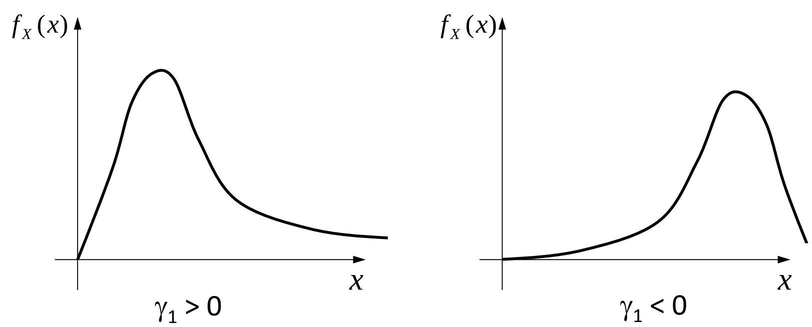
Threrefore the expected value and the variance of a random variable are functions of the first and second moments.

Skewness of a Random Variable

The skewness γ_1 of a random variable measures the extent to which a probability distribution of a real-valued random variable leans to one side of the mean. The skewness is defined as

$$\gamma_1 = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{1}{\sigma^3}\int_{-\infty}^{+\infty}(x-\mu)^3 f_X(x)dx$$

The skewness value can be positive or negative, or even undefined.



Another popular defintion of skewness are the Pearson skewness coefficients, the first coefficient is defined as (Mean-Mode)/Standard Deviation and the second one as (Mean-Median)/Standard Deviation.

Kurtosis of a Random Variable

The fourth standarized moment of a random variable is defined as

$$\beta_2 = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{1}{\sigma^4}\int_{-\infty}^{+\infty}(x-\mu)^4 f_X(x)dx$$

The excess kurtosis defined as

$$\gamma_2 = \beta_2 - 3$$

The 3 comes from the fact that $\beta_2 = 3$ for a Gaussian random variable. A high kurtosis distribution has a sharper peak and longer, fatter tails, while a low kurtosis distribution has a more rounded peak and shorter, thinner tails.

Random variables with zero excess kurtosis are called mesokurtic. The most prominent example of a mesokurtic distribution is the Gaussian distribution family (by definition). A random variable with positive excess kurtosis is called leptokurtic. "Lepto-" means "slender". Examples of leptokurtic distributions include the Student's t-distribution, Rayleigh distribution, Laplace distribution, exponential distribution, Poisson distribution and the logistic distribution. Such distributions are sometimes termed *super-Gaussian*.

A distribution with negative excess kurtosis is called platykurtic. "Platy-" means "broad". In terms of shape, a platykurtic distribution has a lower, wider peak around the mean and thinner tails. The most platykurtic distribution of all is the Bernoulli distribution with $p = \frac{1}{2}$ (for example the number of times one obtains "heads" when flipping a coin once, a coin toss), for which the excess kurtosis is -2. Such distributions are sometimes termed *sub-Gaussian*.