SINE RULE MASTERY

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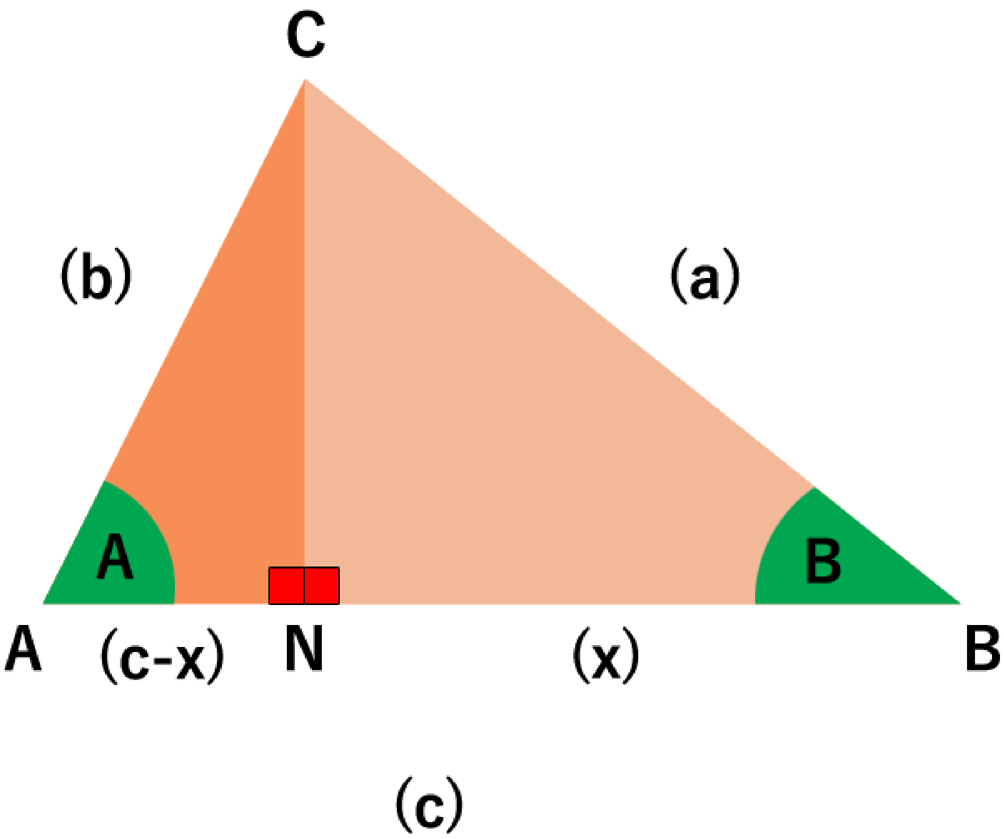
In this document, I'll be demonstrating how to derive the sine rule formula from absolute scratch.

In order to derive the sine rule formula, one must produce a diagram like the one below...



If you look at the diagram above, you will notice that:

1. Angle A resides at point A.
2. Angle B resides at point B.
3. Angle A + Angle B + Angle C = 180 degrees, meaning that Angle C=180 degrees – (Angle A + Angle B)
4. The length of (a) is equal to the length of the line BC.
5. The length of (b) is equal to the length of the line AC.
6. The length of (c) is equal to the length of the line AB.
7. The length of (x) is equal to the length of the line BN.
8. The length of (c-x) is equal to the length of the line AN.

9. The line CN is perpendicular to the line AB.
10. The point N resides on the line AB.
11. The triangle ABC is made up of two right angled triangles, both ACN and BCN.

Now, before we begin deriving the sine rule formula, there are a few other rules I must tell you about:

$$\sin(\theta)=\frac{O}{H}$$

$$\sin(P+Q)=sinPcosQ+cosPsinQ$$

$$\sin(P-Q)=sinPcosQ-cosPsinQ$$

$$Adjacent^2+Opposite^2=Hypotenuse^2$$

Alright, so we've designed a great diagram and we've got all the tools we require to derive the sine rule formula. Let's begin doing some maths...

First of all, let's find the area of the triangle ABC...

Let's say that $A_R=Area$ and that:

$$A_R=(\frac{(c-x)*CN}{2})+(\frac{x*CN}{2})$$

$$A_R=\frac{CN(c-x)+x*CN}{2}$$

$$A_R=\frac{CN*\{(c-x)+x\}}{2}$$

Therefore:

$$A_R=\frac{CN*c}{2}$$

So, with a bit of maths, we've discovered that:

$$A_R = \frac{CN * c}{2}$$

But what is CN?

Well, we know that:

$$\sin(A) = \frac{O}{H} = \frac{CN}{b}$$

Therefore:

$$b * \sin(A) = CN$$

This means we can say that:

$$A_R = \frac{b * \sin(A) * c}{2} = \frac{1}{2} * bc * \sin(A)$$

We also know that:

$$\sin(B) = \frac{O}{H} = \frac{CN}{a}$$

Therefore:

$$a * \sin(B) = CN$$

And this means that:

$$A_R = \frac{a * \sin(B) * c}{2} = \frac{1}{2} * ac * \sin(B)$$

As the two area formulas we've derived are equivalent, we can indeed say that:

$$\frac{bc*\sin(A)}{2}=\frac{ac*\sin(B)}{2}$$

Now, observe what happens when we manipulate both sides of the equation above in the manner below:

$$(\frac{2}{c*\sin(A)*\sin(B)})*(\frac{bc*\sin(A)}{2})=(\frac{2}{c*\sin(A)*\sin(B)})*(\frac{ac*\sin(B)}{2})$$

What do we get?

$$\frac{b}{\sin(B)}=\frac{a}{\sin(A)}$$

So, we've derived the first part of the sine rule formula. Now we have to find a way to reconcile the formula above with $\frac{c}{\sin(C)}$.

Ok, so let's figure out what sinC is exactly...

$$C=180^o-(A+B)$$

Therefore:

$$\sin(C)=\sin(180^o-(A+B))$$

Using the trigonometric identity $\sin(P-Q)=sinPcosQ-cosPsinQ$, we can conclude that:

$$\sin(C)=sin180^o*\cos(A+B)-cos180^o*\sin(A+B)$$

It turns out though that:

$sin180^o = 0$ and $cos180^o = -1$

Therefore:

$$\sin(C) = \sin(A+B)$$

Now:

$$\sin(P+Q) = sinPcosQ + cosPsinQ$$

Therefore sin(C) transforms into:

$$\sin(C) = sinAcosB + cosAsinB$$

However, we know that:

$$\sin(A) = \frac{O}{H} = \frac{CN}{b}$$

$$\sin(B) = \frac{O}{H} = \frac{CN}{a}$$

We can also discover that:

$$\cos(A) = \frac{A}{H} = \frac{c-x}{b}$$

$$\cos(B) = \frac{A}{H} = \frac{x}{a}$$

Therefore:

$$\sin(C)=(\frac{CN}{b})*(\frac{x}{a})+(\frac{(c-x)}{b})*(\frac{CN}{a})$$

$$\sin(C)=(\frac{x*CN}{ab})+(\frac{(c-x)*CN}{ab})$$

$$\sin(C)=\frac{x*CN+CN*(c-x)}{ab}$$

$$\sin(C)=\frac{CN*\{x+(c-x)\}}{ab}$$

$$\sin(C)=\frac{CN*c}{ab}$$

This means that:

$$ab*\sin(C)=CN*c$$

But:

$$A_R=\frac{CN*c}{2}$$

Therefore:

$$CN*c=2\mathrm{A}_R$$

And as a result:

$$2\mathrm{A}_R=ab*\sin(C)$$

Which means that:

$$A_R = \frac{1}{2} * ab * \sin(C)$$

From here we could say that:

$$\frac{ab*\sin(C)}{2} = \frac{bc*\sin(A)}{2}$$

And if we manipulate the equation above in the manner below:

$$\left(\frac{2}{b*\sin(A)*\sin(C)}\right)*\left(\frac{ab*\sin(C)}{2}\right) = \left(\frac{2}{b*\sin(A)*\sin(C)}\right)*\left(\frac{bc*\sin(A)}{2}\right)$$

We'll get:

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

As:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

We can now say that:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

And we've derived the sine rule formula. Congratulations, you are now a sine rule master!

For more proofs like these, please visit my website http://www.mathsvideos.net.

Kind regards,

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