COSINE RULE MASTERY

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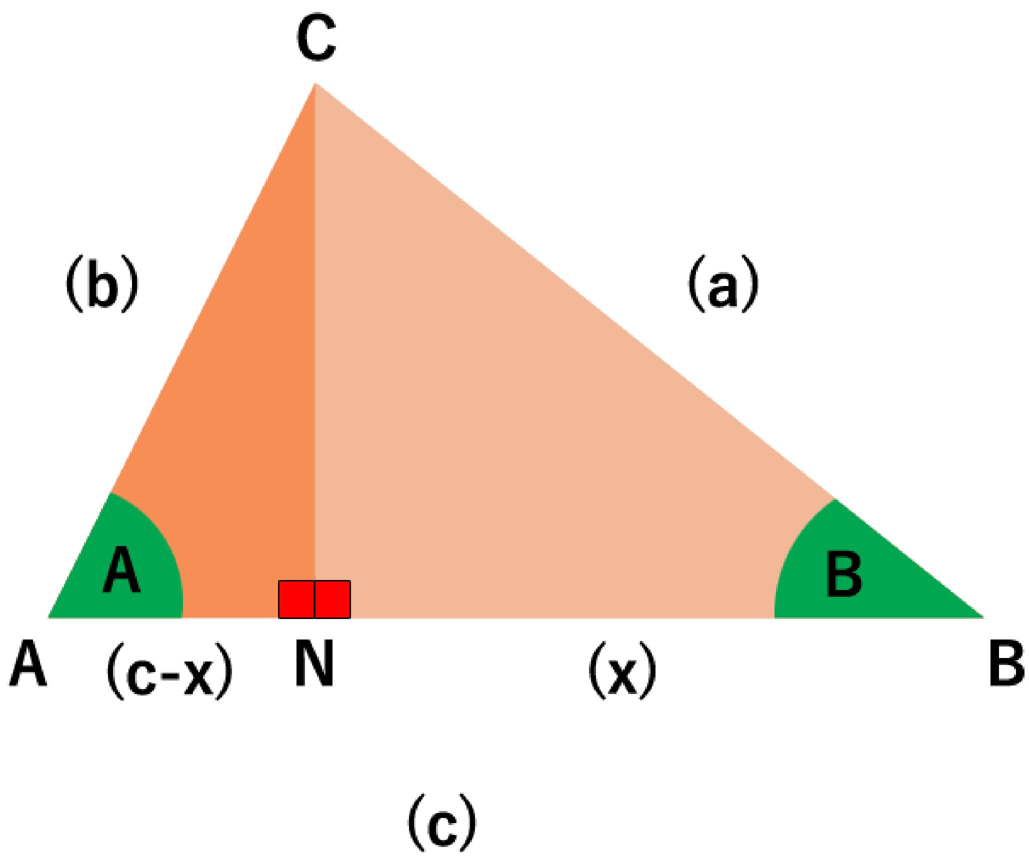
In order to produce the 3 main cosine rule formulas, you must produce the diagram below:



Looking at the diagram above, what you must acknowledge is the fact that:

1. A is a sharp edge with a point A and has an angle A.
2. B is a sharp edge with a point B and has an angle B.
3. C is a sharp edge with a point C and has an angle C.

You must also notice that the triangle ABC was formed using two right angled triangles, both ACN and BCN respectively.

Furthermore, the line CN is perpendicular to the line AB.

The point N sits on the line AB.

What you should also acknowledge is the fact that:

1. The length (a) is the length of the line BC.
2. The length (b) is the length of the line AC.
3. The length (c) is the length of the line AB.
4. The length (x) is the length of the line BN.
5. The length (c-x) is the length of the line AN.

With these facts in mind, we can now find the 3 main cosine rule formulas. With the use of the SOH CAH TOA rule, we'll be able to find the 3 main cosine rule formulas within a very short period of time.

Let's remind ourselves precisely what the SOH CAH TOA rule actually is...

$$\sin(\theta)=\frac{O}{H}, \cos(\theta)=\frac{A}{H}, \tan(\theta)=\frac{O}{A}$$

Let's also give credence to Pythagoras's formula:

$$Adjacent^2+Opposite^2=Hypotenuse^2$$

And finally, we are going to have to use two important trigonometric identities to find the cosine of the angle C.

$$\cos(A-B)=cosAcosB+sinAsinB$$

$$\cos(A+B)=cosAcosB-sinAsinB$$

Ok, so now that we've got the tools we need to find the 3 main cosine rule formulas, let's do some mathematics...

Using Pythagoras's formula, let's state that:

$$(c-x)^2+CN^2=b^2$$

Therefore:

$$CN^2=b^2-(c-x)^2$$

Also:

$$x^2+CN^2=a^2$$

Therefore:

$$CN^2=a^2-x^2$$

And this means that:

$$a^2-x^2=b^2-(c-x)^2$$

$$a^2-x^2=b^2-\{c^2-2\mathrm{cx}+x^2\}$$

$$a^2-x^2=b^2-c^2+2\mathrm{cx}-x^2$$

Therefore:

$$a^2=b^2-c^2+2\mathrm{cx}$$

And this is a formula we'll be using on more than one occasion when trying to figure out the 3 main cosine rule formulas.

Ok, so we have the formula:

$$a^2 = b^2 - c^2 + 2\text{cx}$$

But what is the value of x? Can we find it?

Of course... It turns out that:

$$\cos(B) = \frac{A}{H} = \frac{x}{a}$$

This means that:

$$a * \cos(B) = x$$

Now let's plug $a * \cos(B) = x$ into the formula $a^2 = b^2 - c^2 + 2\text{cx}$ to see what we get:

$$a^2 = b^2 - c^2 + 2\text{c}(a * \cos(B))$$

And moving on from here we get:

$$a^2 = b^2 - c^2 + 2\text{ac} * \cos(B)$$

Which means that:

$$b^2 = a^2 + c^2 - 2\text{ac} * \cos(B)$$

So, we've now discovered one of the main cosine rule formulas. Now we must find the other two...

Now, using the formula $a^2=b^2-c^2+2cx$ once again, let's find one more cosine rule...

Does x have another value? The answer to this question is indeed 'yes'...

It turns out that:

$$\cos(A)=\frac{A}{H}=\frac{(c-x)}{b}$$

This means that:

$$b*\cos(A)=c-x$$

And as a result:

$$x=c-b*\cos(A)$$

Plugging $x=c-b*\cos(A)$ into the formula $a^2=b^2-c^2+2cx$ we get:

$$a^2=b^2-c^2+2c*\{c-b*\cos(A)\}$$

Which leads to the conclusion that:

$$a^2=b^2-c^2+2c^2-2bc*\cos(A)$$

$$a^2=b^2+c^2-2bc*\cos(A)$$

And there we have it; we've found the second main cosine rule formula.

Now, to complete our mission we must find our last main cosine rule formula. This task isn't going to be too easy, however, with a bit of patience we'll be finding our last formula in practically a few minutes.

Ok, let's think about what the angles A, B and C add up to...

It turns out that:

$$A+B+C=180^o$$

This means that:

$$C=180^o-A-B$$

Therefore:

$$C=180^o-(A+B)$$

So, we know angle C's relationship to both angle A and B. We can therefore say that:

$$\cos(C)=\cos(180^o-(A+B))$$

Using the trigonometric identity $\cos(A-B)=cosAcosB+sinAsinB$, we can transform the equation above into:

$$\cos(C)=cos180^o*\cos(A+B)+sin180^o*\sin(A+B)$$

But, it turns out that:

$sin180^o=0$ and $cos180^o=-1$

So, we get:

$$\cos(C)=-\cos(A+B)$$

Using the trigonometric identity $\cos(A+B)=cosAcosB-sinAsinB$, we can conclude that:

$$\cos(C)=-\{cosAcosB-sinAsinB\}$$

Therefore:

$$\cos(C)=sinAsinB-cosAcosB$$

Ok, so we now have an interesting equation to work with. We already know that:

$\cos(A)=\frac{A}{H}=\frac{(c-x)}{b}$ and $\cos(B)=\frac{A}{H}=\frac{x}{a}$

This means that we must find sinA and sinB to produce the last cosine formula. This task should be fairly easy...

$$\sin(A)=\frac{O}{H}=\frac{CN}{b}$$

$$\sin(B)=\frac{O}{H}=\frac{CN}{a}$$

From here, we can say that:

$$\cos(C)=(\frac{CN}{b})*(\frac{CN}{a})-(\frac{(c-x)}{b})*(\frac{x}{a})$$

And this equation can be changed into:

$$\cos(C)=(\frac{CN^2}{ab})-(\frac{x(c-x)}{ab})$$

Which can be further transformed into:

$$\cos(C)=\frac{CN^2-x(c-x)}{ab}$$

If we continue transforming the equation above, we get:

$$ab*\cos(C)=CN^2-x(c-x)$$

$$ab*\cos(C)=CN^2-cx+x^2$$

However, we know that $x^2+CN^2=a^2$, which means that the equation above changes into:

$$ab*\cos(C)=a^2-cx$$

Now we must get rid of the x in the equation above to get the last cosine rule formula we were looking for...

It turns out that:

$$a^2=b^2-c^2+2\text{cx}$$

Which means that:

$$2\text{cx}=a^2+c^2-b^2$$

And this ultimately means that:

$$cx=\frac{a^2+c^2-b^2}{2}$$

So, we can now produce the equation:

$$ab*\cos(C)=a^2-(\frac{a^2+c^2-b^2}{2})$$

Multiply both sides of the equation above by 2 and we get:

$$2ab*\cos(C)=2a^2-(a^2+c^2-b^2)$$

$$2ab*\cos(C)=2a^2-a^2-c^2+b^2$$

Therefore:

$$2ab*\cos(C)=a^2-c^2+b^2$$

Which ultimately means that:

$$c^2=a^2+b^2-2ab*\cos(C)$$

This is the last cosine rule formula we were looking for. Congratulations in finding it!

You are now a cosine rule master. :-)

You can prove that:

$$a^2=b^2+c^2-2bc*\cos(A)$$

$$b^2=a^2+c^2-2ac*\cos(B)$$

$$c^2=a^2+b^2-2ab*\cos(C)$$

For more proofs like these, please visit my maths website http://www.mathsvideos.net. I hope you've enjoyed reading this document.

Kind regards,

Thiago De Carvalho